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Market-based allocation mechanisms for lot-size decision makers and electric power utilities

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Market-based allocation mechanisms for lot-size decision makers and electric power utilities

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This dissertation is dedicated to my parents and my wife.

GENERAL INTRODUCTION

General Background and Objectives

In this dissertation, we examine how lot-size decision makers and electric power utilities determine critical economic quantities (e.g., the order quantities for lot-size decision makers and transmission service chargers for electric power utilities) so as to improve the economic efficiency of operations. Throughout this dissertation, the optimal policies are obtained through linear and nonlinear programming techniques. For each model, interesting managerial insights and economic implications are obtained and illustrative numerical examples are provided.

For lot-size decision makers, we extend the traditional economic order quantity model by considering various aspects of model environments such as inventory/pricing policies and different performance criteria (profit maximization vs. return on investment maximization). By analyzing the optimal solutions derived in our models, several interesting managerial insights are obtained. On the other hand, for electric power utilities, we propose a two-stage trilateral brokerage system for electric power transactions by considering the costs and benefits to buyers, sellers, and intermediate transmission utilities. By employing economic analysis and linear and nonlinear programming techniques, we show that significant gains in economic efficiency (often measured in terms of cost saving) can be achieved. Details of background and motivation for our study (first for the lot-size decision makers, then for the electric power utilities) are as follows.

Keeping an inventory to meet potential demand in the future is prevalent in most businesses. Manufacturers, wholesalers, and retailers generally have a stock of goods on hand. How to determine the "inventory policies" (i.e., when and how much to order/produce as well as how much to charge per unit) becomes a critical issue for lot-size decision makers. A simple model representing production-inventory situation is given by the well-known traditional economic order quantity (EOQ) model (see e.g., Hillier and Lieberman, 1995).

The traditional EOQ model determines the production-inventory system by considering only cost factors consisting of a fixed setup cost, a variable unit production cost, and an inventory holding cost. It should be pointed out, however, that the inventory policies of numerous businesses may depend on its relations to other business policies regarding pricing and sales. In this study, we attempt to integrate the policies of inventory and pricing/sales so as to maximize the decision maker's benefit.

The optimal inventory policies under price changes, based on the classical economic order quantity (EOQ) models, have been extensively studied (see e.g., Goyal, Srinivasan, and Arcelus, 1991, Lev and Weiss, 1990, Ardalan, 1988, 1991, 1994, and Aull-Hyde, 1992, etc.). In their papers, the wide range of industrial practices and applicability of price changes are discussed in details. Inventory policies under disposal options have also been studied to some extent (see e.g., Rosenfield, 1989, Sethi, 1984, and Tersine and Toelle, 1984, etc.).

The numerous studies of these topics in the literature reflect the relevance and importance of the topics to both academicians and practitioners. Also, it is intuitive that, given a temporary sale, a buyer may find it beneficial to place a special order at a reduced price and/or dispose some of his on-hand inventory at a salvage value because these transactions may result in reduced total cost for the inventory system. Up to now, however, there have been few analytical models that integrates inventory and disposal policies under temporary sales. Hence, it is highly desirable to construct and analyze quantitative models of inventory and disposal policies under temporary sales.

First, we investigate the optimal inventory and disposal policies for a buyer who is just informed of a temporary sale by his supplier. It is shown how the buyer determines the optimal inventory and disposal quantities so as to exploit the temporary sale.

This inventory model is extended by focusing on the period between the announcement and commencement of a sale. By analyzing the optimal solutions for this extended model, it is shown how the pre-announcement can be utilized to maximize cost saving.

Next, we examine an inventory and investment in setup operations model under profit maximization and under return on investment maximization. From the optimality conditions, the optimal order quantity, investment level, and several interesting managerial insights are obtained.

Finally, we consider a published multi-product EOQ model with constraints, and examine its optimal inventory and pricing policies. We show that there are two critical errors, and provide correct design and analysis by re-formulating and re-solving the entire model.

For electric power utilities, in the United States, they are currently facing a drastic transformation from traditional, regulated, and vertically integrated environments to de-regulated and competitive environments (see e.g., McCalley and Sheble, 1994). A primary motivation for this transformation is to improve the economic efficiency in the electric power industry. A critical research area where the electric power industry can improve the economic efficiency is that of power interchange in an interconnected power system. The power interchange may improve the economic efficiency because there exist some potential savings whenever the difference in incremental production costs among utilities is significant and some extra production capacities exist.

In this dissertation, we propose a two-stage trilateral (buyer, seller, and intermediate transmission utility) brokerage system for power transactions. In the first stage, a linear programming model is proposed to match bids from potential buyers and sellers. In the second stage, hierarchical criteria (such as the number of intermediate transmission utilities involved) are employed to determine the transmission routes based on the transmission costs to the intermediate transmission utilities.

Finally, we extend the two-stage trilateral brokerage system by allowing multiple bids from potential buyers and sellers, and by proposing a nonlinear programming model for transmission route selection. By employing economic analysis at each stage, we show that significant gains in economic efficiency can be achieved.

Dissertation Organization

This dissertation is composed of six papers which may be suitable for publication. In particular, the first paper "OPTIMAL INVENTORY AND DISPOSAL POLICIES IN RESPONSE TO A SALE" is accepted and will appear in *International Journal of Production Economics*. The fourth paper "A MULTI-PRODUCT EOQ MODEL WITH PRICING CONSIDERATION -- T. C. E. CHENG'S MODEL REVISITED" appears in *Computers and Industrial Engineering: An International Journal*, Volume 26, Number 4, Page 787-794, 1994. The fifth paper "A TWO-STAGE BROKERAGE SYSTEMS FOR ELECTRIC POWER TRANSACTIONS" is presented at the Fourth Industrial Engineering Research Conference, Nashville, Tennessee, May 1995, and will appear in the *Proceedings of the Fourth Industrial Engineering Research Conference*.

Also, the second paper "OPTIMAL INVENTORY POLICIES IN RESPONSE TO A PRE-ANNOUNCED SALE" is to be submitted to *IIE Transactions*. The third paper "OPTIMIZATION CRITERIA FOR INVENTORY-INVESTMENT IN SETUP OPERATIONS POLICIES: PROFIT VS. RETURN ON INVESTMENT" is to be submitted to *Decision Sciences*. And the sixth paper "A TRILATERAL BROKERAGE SYSTEM FOR POWER TRANSACTIONS" is to be submitted to *International Journal of Energy Research*.

In Chapter 1 "OPTIMAL INVENTORY AND DISPOSAL POLICIES IN RESPONSE TO A SALE", we construct and analyze an EOQ-type model for a buyer who is just informed of a temporary sale. The buyer is assumed to have an option to place special orders and an option to dispose some of his on-hand inventory. The key feature differentiating our model from the extant literature on inventory models is that the optimal inventory and disposal policies are fully integrated and simultaneously determined. The optimal policies are derived in closed-form from comparing cost savings of various cases of strategies, and several interesting managerial insights are obtained by analyzing the closed-form optimal policies.

In Chapter 2 "OPTIMAL INVENTORY POLICIES IN RESPONSE TO A PRE-ANNOUNCED SALE", we construct and analyze an EOQ-type model for a buyer who is just informed of a pre-announced sale. By "a pre-announced sale", we mean the announcement time of the sale occurs before the beginning time of the sale. Under the pre-announced sale, the buyer is assumed to have an option to adjust his replenishment strategy before the sale is effective and an option to place special orders during the temporary sale. For such a buyer, optimal inventory policies are derived by comparing cost savings of various cases. By analyzing the optimal inventory policies, several managerial insights are obtained. For example, as the period between the announcement time of the sale and the commencement of the sale increases, the optimal cost saving will increase or remain the same. In addition, as the duration of the sale increases, the optimal cost saving will increase or remain the same.

In Chapter 3 "OPTIMIZATION CRITERIA FOR INVENTORY-INVESTMENT IN SETUP OPERATIONS POLICIES: PROFIT VS. RETURN ON INVESTMENT", we construct and analyze optimal policies for inventory and investment in setup operations under profit maximization and under return on investment maximization. Under a general functional form of investment in setup operations, we derive the optimality conditions under profit maximization and under return on investment maximization. By comparing and contrasting the optimality conditions, several interesting economic implications are obtained. Also, for two specific functional forms of investment in setup operations (linear and hyperbolic), the closed-from optimal solutions and the decision making rules are derived. From the solution and rules, additional economic implications are obtained.

In Chapter 4 "A MULTI-PRODUCT EOQ MODEL WITH PRICING CONSIDERATION -- T. C. E. CHENG'S MODEL REVISITED", we present two major revisions/ corrections regarding a recent paper by T. C. E. Cheng (1990). First, we note that a critical assumption of the equal replenishment cycle length for all products is stated, but not incorporated into the mathematical formulation in Cheng (1990). In this paper, we re-formulate the problem with the equal replenishment cycle length incorporated and derive the corresponding Kuhn-Tucker optimality conditions. Next, under the linear demand assumption, we show that the closed-form solutions provided by Cheng (1990) may result in non-optimal solutions. The reason is that Cheng (1990) failed to derive conditions under which the closed-form solutions may be optimal. In this paper, by employing the trigonometric methods (see e.g., Porteus, 1985), we derive the optimal closed-form solution that is unique and obtain the conditions under which the optimal closed-form solution is valid.

In Chapter 5 "A TWO-STAGE BROKERAGE SYSTEM FOR ELECTRIC POWER

TRANSACTIONS", we propose a two-stage brokerage system for electric power transactions. At the first stage of the brokerage system, a linear programming model is set up to maximize the total saving in matching bids from buyers and sellers. At the second stage of the brokerage system, how to determine the route(s) to transmit the transacted power is investigated. By employing economic analysis at each stage, we show that significant gains in economic efficiency can be achieved.

In Chapter 6 "A Trilateral Brokerage System for Power Transactions", we extend the two-stage trilateral brokerage system for electric power transactions discussed in Chapter 5 to the following two aspects. First, multiple purchase bids and multiple sale bids from each buyer and seller are allowed in this paper. By formulating a linear program to match bids from sellers and buyers, we show that the total cost saving can be significantly improved. Second, instead of employing the pre-specified rules proposed in Chapter 5, we mathematically formulate the problem of selecting routes to transmit the transacted power as a nonlinear program and obtain the corresponding optimal solution. By incorporating the above two aspects and by employing a numerical example, we show that the economic efficiency of the brokerage system for power transactions can be significantly improved.

The rest of this dissertation is organized as follows. First, those six papers mentioned earlier will be presented sequentially. Next, the general conclusion about this dissertation follows the sixth paper. Finally, the literature cited in the general introduction and the general conclusion are listed.

CHAPTER I.

OPTIMAL INVENTORY AND DISPOSAL POLICIES IN RESPONSE TO A SALE

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ABSTRACT

We construct and analyze an EOQ-type model for a buyer who is just informed of a temporary sale. The buyer is assumed to have an option to place special orders and an option to dispose some of his on-hand inventory. The key feature differentiating our model from the extant literature on inventory models is that the optimal inventory and disposal policies are fully integrated and simultaneously determined. The optimal policies are derived in closed-form from comparing cost savings of various cases of strategies, and several interesting managerial insights are obtained by analyzing the closed-form optimal policies.

1. INTRODUCTION

In this paper, an EOQ-type model is constructed and analyzed for a buyer who is just informed of a temporary sale. Under the temporary sale, the buyer is assumed to have an option to place special orders and an option to dispose some of the on-hand inventory. By comparing cost savings of various cases of strategies (see e.g., Tersine [1]), we obtain the closed-form solutions of the optimal inventory and disposal policies. These inventory and disposal policies are fully integrated and simultaneously determined. By analyzing the closed-form optimal policies, we obtain interesting managerial insights for the buyer.

The optimal inventory policies under price changes (increases or decreases), based on the classical economic order quantity (EOQ) models, have been extensively studied (see e.g., Lev and Weiss [2]). Also, for temporary price discount, there have been numerous studies investigating the optimal replenishment and inventory policies (see e.g., Ardalan [3]). Aucamp and Kuzdrall [4] [5] focus on one-time-only sales and determine the optimal special order quantities by employing a discounted cash flow approach. Ardalan [6] deals with a temporary price discount and derives the optimal inventory policies by employing a net present value method and/or by incorporating the marketing effect on demand. Aull-Hyde [7] discusses the optimal ordering rules in response to supplier restrictions on special order sizes that accompany temporary price decreases. In Tersine and Barman [8], a composite EOQ model, which can be disaggregated into several traditional EOQ models, is developed to determine the optimal levels of order quantity and backorder quantity in response to a temporary price discount. We note that the models constructed and analyzed in the last three papers assume that the sale period is short relative to the regular EOQ replenishment cycle and the sale period is within a regular EOQ replenishment cycle. On the other hand, the optimal replenishment strategies for any length of sale time horizon have also been investigated by a number of researchers (see e.g., Goyal [9] and Tersine and Schwarzkopf [10]).

Inventory policies with disposal options have also been extensively studied. Rosenfield [11] analyzes the costs of holding and disposing of slow-moving inventory under stochastic demand and perishing. Sethi [12] presents an optimal inventory and disposal model for a buyer faced with all-unit quantity discounts offered by a seller. Tersine and Toelle [13] develops models to determine how much stock should be retained and how much should be disposed of when an excess inventory of that item currently exists. In their paper, a list of eight reasons for excess inventory is provided. The numerous studies of these two topics in the literature reflect the relevance and importance of the topics to both academicians and practitioners. Also, it is intuitive that, given a temporary sale, a buyer may find it beneficial to place special orders at a reduced price and/or dispose some of on-hand inventory at a salvage value because these transactions may result in reduced total cost for the inventory system. Up until now, however, there have been few analytical models that integrate inventory and disposal policies under temporary sales. Hence, considering the fact that numerous firms utilize EOQ-based decision making processes for such policies (see e.g., Tersine and Toelle [13]), it is highly desirable to construct and analyze EOQ-based models of inventory and disposal policies under temporary sales.

In this paper, we will focus on optimal inventory and disposal policies for a buyer who is just informed of a temporary sale. By "just informed," we mean that the buyer is able to place special orders and make disposals from that time point on. That is, the emphasis is on **when** the buyer is **able to respond** to a sale. Hence, if the buyer is able to respond to a sale from a particular time point on due to administrative, informational, organizational, and/or other reasons, that particular time point is viewed as the time point at which the buyer is "just informed". In addition, by "a temporary sale," we mean that the sale period is short relative to the regular EOQ replenishment cycle. Specifically, we will restrict our attention to the case that the sale period is less than one regular EOQ replenishment cycle. We note that the sale period could actually be quite long in absolute duration (e.g., 3 months) when the regular EOQ replenishment cycle is also long in absolute duration (e.g., 6 months). Hence, this assumption is not as restrictive as it may first appear and such an assumption can be found in several publications (see e.g., Ardalan [3] [6], Aull-Hyde [7], etc.).

The rest of this paper is organized as follows. We first introduce the model environments and the structure of optimal inventory and disposal policies. Next, we obtain the closed-form optimal solutions by comparing the cost saving of various cases. We then present the decision process for the optimal inventory and disposal policies and provide illustrative numerical examples. From the numerical results, several managerial insights and properties are derived. Finally, we summarize and comment on further research.

2. MODEL ENVIRONMENTS

2.1 Assumptions and Definitions

In our model, a buyer determines the optimal order quantity from his supplier based on the classical EOQ model. As in numerous EOQ-type models, we make the following assumptions.

1) the buyer's demand is constant over time,

2) no shortage is allowed,

3) replenishment is instantaneous,

4) lead time is zero.

We note that, the assumption of zero lead time is made for simplicity and a positive lead time can be easily incorporated into our model. Also, the following definitions of the classical EOQ model are employed.

R: the buyer's demand per unit time (e.g., annual demand).

P: the purchase price per unit to the buyer from the supplier before and after the sale.

F: the holding cost per unit time as a fraction of the unit purchase price.

C: the ordering (setup) cost per order (i.e., a fixed cost independent of the order quantity).

Q_0: the economic order quantity given the purchasing price per unit, P. i.e., $Q_0 = (\frac{2CR}{PF})^{0.5}$.

We also note that the inventory holding cost per unit time F is assumed to be a fraction of the original unit purchase price not the current or future unit price.

Let us suppose that the buyer, at time point t_b, is informed that there is a sale effective now through time point t_e, and the buyer is expected to make his decisions regarding his inventory and disposal policies. As mentioned earlier, we also assume that the sale period is less than one regular EOQ replenishment cycle (i.e., $t_e - t_b < (\frac{2C}{PFR})^{0.5}$).

We will denote the magnitude of price decrease in the sale by d ($d > 0$), and the new purchasing price per unit for the buyer will be $P - d$. Let us assume that the buyer has *an option to instantaneously dispose* any inventory at a salvage value of S per unit, where $P - d > S$. $P - d > S$ is assumed so as to exclude the possibility of arbitrages. Let us also assume that the buyer has *an option to place special orders* during the sale, at the reduced price of $P - d$ per unit, regardless of the on-hand inventory level. Given these two options, the buyer must determine the optimal inventory and disposal policies. In response to a sale, a special order at the decreased price ($P-d$) and/or a disposal at the salvage value of S during the sale can be beneficial to the buyer because these transactions may result in reduced inventory holding cost components (such as capital costs, insurance costs, and taxes). In order to investigate the optimal inventory and disposal policies for the buyer, we introduce the following additional definitions.

q: the level of inventory (stock position) at time point t_b.

K: the disposal setup cost.

We note that, for our model, we will optimally determine the special order quantities and the disposal quantities as well as the time points at which special orders are placed and disposals occur. Finally, throughout the rest of the paper, we will assume that the products are withdrawn from inventory on a first-in, first-out (FIFO) basis. This is a reasonable assumption in numerous practical inventory systems, and it facilitates tractable construction and analysis of the model.

2.2 The Structure of an Optimal Policy

Given the fact that the buyer is informed of the sale, the special orders and disposals can be viewed as useful tools to reduce the total costs of operation. In this subsection, we investigate the special orders and disposals with respect to quantity and time. Specifically, we will initially assume that there will be only one special order and one disposal and derive interesting properties of the optimal policy. Based on these interesting properties, we will examine multiple special orders and disposals. Such an investigation will result in simplification of the mathematical models for the problem.

Let us denote a special order quantity and a disposal quantity during the sale by Q_s and D, respectively. Also, we define x to be the time interval between t_b and the time point at which the disposal occurs. In addition, we define y to be the time interval between t_b and the time point at which the special order occurs. Furthermore, we denote the inventory level (including the remnant inventory) after the special order is received at time point (t_b+y) by Q_z. Figure 1 illustrates two possible policies for the buyer to follow. One is to dispose D units of on-hand

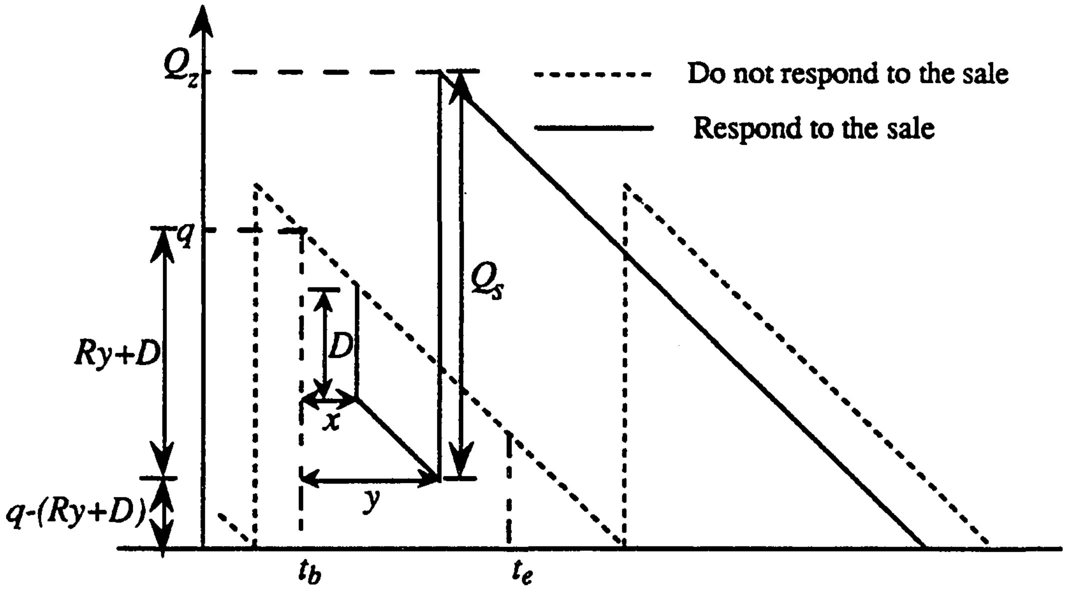

Figure 1. General inventory behavior with options to make a disposal and place a special order

inventory at time point (t_b+x) and to place a special order at time point (t_b+y). We will call this policy the "Response" policy. The other one is to ignore both the options to dispose and to place a special order. We call this policy the "Non-Response" policy. In order to measure the cost saving of the "Response" policy over the "Non-Response" policy accurately, the total costs of these two possible policies will be calculated from the time point t_b to the time point $(t_b+y+\frac{Q_z}{R})$ (see e.g., Tersine [1]). The total cost from the time point t_b to the time point $(t_b+y+\frac{Q_z}{R})$ for the "Response" policy, TC_R, can be expressed as follows.

$$TC_R = \frac{q+(q-Rx)}{2}xPF + K - DS + \frac{(q-Rx-D)^2}{2R}PF + C + (Q_z-q+Ry+D)(P-d) + \frac{(P-d)F}{2R}[Q_z^2-(q-Ry-D)^2] \qquad (1)$$

The corresponding total cost for the same duration for the "Non-Response" policy, TC_{NR}, is given by

$$TC_{NR} = \frac{q^2PF}{2R} + \frac{Q_z-q+Ry}{R}(PR+(2CRPF)^{0.5}) \qquad (2)$$

From the relations (1) and (2), the cost saving of the "Response" policy over the "Non-Response" policy, CS, is given by $CS = TC_{NR} - TC_R$. The objective now is to find the optimal x, y, and Q_z, which will maximize CS. Namely,

$$\underset{x,y,Q_z}{\text{Maximize}}\ CS = TC_{NR} - TC_R \qquad (3)$$

From the maximization of the above problem, the following first derivatives can be easily obtained.

$$\frac{\partial CS}{\partial Q_z} = \frac{PR+(2CRPF)^{0.5}}{R} - (P-d) - \frac{(P-d)FQ_z}{R} \qquad (4)$$

$$\frac{\partial CS}{\partial x} = -DPF \qquad (5)$$

$$\frac{\partial CS}{\partial y} = (P-d)F\left[\frac{(2CRPF)^{0.5}+dR}{(P-d)F} - (q-Ry-D)\right] \tag{6}$$

By setting equation (4) equal to zero, the optimal Q_z can be obtained as follows

$$Q_z = \frac{(2CRPF)^{0.5}+dR}{(P-d)F} = \frac{(2CRPF)^{0.5}+PR}{(P-d)F} - \frac{R}{F} = \frac{Rd}{(P-d)F} + \frac{P}{P-d}Q_o \tag{7}$$

We note that the expression of Q_z in equation (7) is identical to the special order quantity shown in Tersine [1] when on-hand inventory level is zero. By substituting equation (7) into equation (6), we have the following expression for $\frac{\partial CS}{\partial y}$.

$$\frac{\partial CS}{\partial y} = (P-d)F[Q_z - (q-Ry-D)] > 0 \tag{8}$$

We summarize those results in the following two propositions.

<u>Proposition 1.</u> Assume that the buyer makes a disposal of on-hand inventory during the sale. Then, $\frac{\partial CS}{\partial x} < 0$.

Proposition 1 implies that the cost saving will increase when x is decreased. That is, if the buyer makes a disposal, his optimal strategy is to dispose as early as possible (i.e., dispose at time point t_b when $x = 0$). From the fact that $\frac{\partial CS}{\partial x} < 0$, for the case of one disposal, it can be easily shown that the strategy of multiple disposals during the sale period is never optimal.

<u>Proposition 2.</u> Assume that the buyer places a special order during the sale. Then, we have:

1) $Q_z = \frac{(2CRPF)^{0.5}+dR}{(P-d)F}$;
2) and $\frac{\partial CS}{\partial y} > 0$.

The economic implications of Proposition 2 are as follows. If the buyer places a special order during the sale, the optimal strategy is to replenish the inventory up to the level $Q_z = \frac{(2CRPF)^{0.5}+dR}{(P-d)F}$ regardless of the level of on-hand inventory.

In addition, (from $\frac{\partial CS}{\partial y} > 0$), the cost saving will increase when y is increased. That is, if the buyer places a special order during the sale, his optimal strategy is to place the special order as late as possible. We note that this observation is consistent with the Theorem 1 in Ardalan [3]. Also, this observation directly leads us to the following conclusion regarding multiple special orders.

Let us first consider the case where the level of on-hand inventory is non-negative at t_e without any special order. If the buyer places a special order, the optimal time point to do so is at time point t_e because $\frac{\partial CS}{\partial y} > 0$. Hence, it can be easily shown that the strategy of multiple special orders during the sale is never optimal.

For the case where the level of on-hand inventory reaches zero before t_e, let us denote the time point at which inventory reaches zero during the sale by t_0 (i.e., $t_0 < t_e$). According to the Theorem 1 in Lev and Weiss [2], we note that the buyer can have a special order Q_z right at t_0 or have some equal-size orders to meet the demand from t_0 to t_e and then place a special order Q_z at t_e (see Figure 2). The following proposition determines the possible optimal inventory strategies for the buyer from time point t_0 to time point t_e.

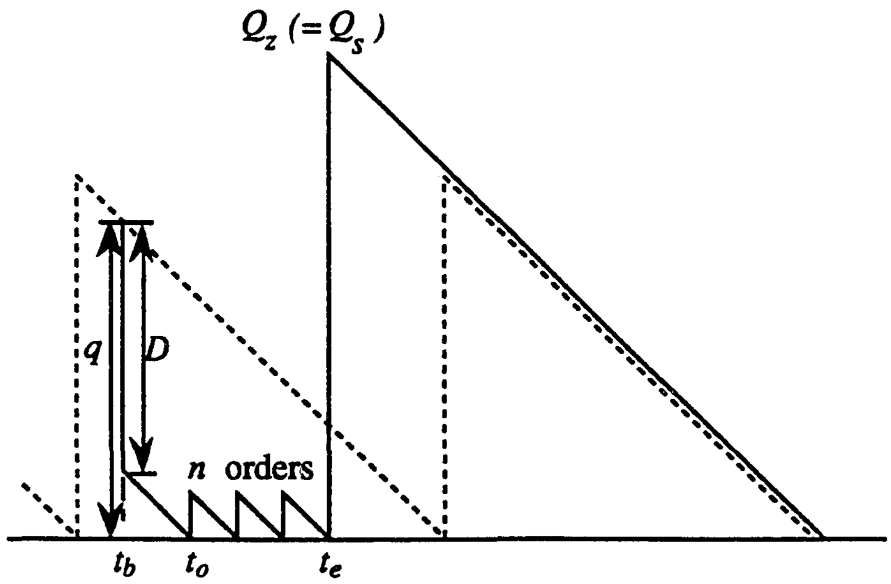

Figure 2. Optimal inventory behavior from t_o to t_e

<u>Proposition 3.</u> Assume that, during the sale, the inventory reaches zero at time point t_o ($t_o < t_e$). The possible optimal strategy for the buyer from t_o to t_e is either to place a special order Q_z at t_o or to place a special order to meet the exact demand from t_o to t_e and an additional special order Q_z at t_e.

<u>Proof:</u>

We note that the buyer minimizes the total cost incurred from t_o to t_e over the number of orders, n. Hence, we have the following total cost minimization objective function.

$$\underset{n}{\text{Minimize}}\ TC_{oe} = nC + \frac{(t_e - t_o)^2 R(P-d)F}{2n} + (t_e - t_o)R(P-d) \tag{9}$$

By setting the first derivative of TC_{oe} with respect to n equal to zero, the optimal number of orders $\hat{n}$ is given by

$$\hat{n} = \frac{(t_e - t_o)}{(2C/((P-d)FR))^{0.5}} \tag{10}$$

It can be easily verified that $0 \leq \hat{n} \leq 1$. By incorporating the integer constraint on the decision variable n, we note that the optimal integer number of orders n^* is equal to 0 or 1. If $n^* = 0$, the buyer places a special order Q_z at t_o. On the other hand, if $n^* = 1$, the buyer has only one order of $R(t_e - t_o)$ at t_o to meet the demand from t_o to t_e and then places a special order Q_z at t_e. Throughout the rest of this paper, we will denote the special order quantity which satisfies the demand from t_o to t_e by Q_s^1. Therefore, for the case that the inventory level reaches zero before t_e, if the buyer places special orders, then the number of special orders during the sale is either one or two.

So far, we have presented the potential structure of an optimal inventory and disposal policies. Under the assumption that the sale period is less than one regular EOQ replenishment cycle, we note that it is possible to have no regular EOQ replenishment point or only one regular EOQ replenishment point during sale period. The following two sections will discuss these two scenarios and derive the corresponding closed-form solutions for the optimal inventory and disposal policies.

3. NO REGULAR EOQ REPLENISHMENT POINT DURING THE SALE ($q \geq R(t_e - t_b)$)

3.1 Description of Exclusive and Exhaustive Cases

In this section, we consider the case that no regular EOQ replenishment point exists during the sale (i.e., $q \geq R(t_e - t_b)$). According to Propositions 1, 2, and 3 in the previous section, the feasible policies can be classified into the following nine mutually exclusive and exhaustive cases.

Case 1): $q > D > 0$, $(q - D) = R(t_e - t_b)$ and $Q_s > 0$ at t_e.

Case 2): $q > D > 0$, $(q - D) > R(t_e - t_b)$ and $Q_s > 0$ at t_e.

Case 3): $q > D > 0$, $(q - D) > R(t_e - t_b)$ and $Q_s = 0$ at t_e.

Case 4): $q > D > 0$, $(q - D) < R(t_e - t_b)$ and $Q_s > 0$ at t_o.

Case 5): $q > D > 0$, $(q - D) < R(t_e - t_b)$, $Q_s^1 = R(t_e - t_o)$ at t_o, and $Q_s > 0$ at t_e.

Case 6): $D = q$, $Q_s > 0$ at t_b.

Case 7): $D = q$, $Q_s^1 = R(t_e - t_b)$ at t_b, and $Q_s > 0$ at t_e.

Case 8): $D = 0$, $Q_s > 0$ at t_e.

Case 9): $D = 0$ and $Q_s = 0$ (i.e. "Non-Response" policy).

Given the above nine cases, we will employ Case 9 of no-special-order and no-disposal (i.e., wait until the remnant inventory is depleted and then purchase $Q_0 = (\frac{2CR}{PF})^{0.5}$ for all subsequent orders) as the benchmark (see e.g., Tersine [1]). Cases 1 through 8 will be examined against this benchmark to determine the optimal disposal amount at time point t_b.

3.2 Cost Saving Comparisons for the case $q \geq R(t_e - t_b)$

In this section, we will examine the cost savings of Case 1 through Case 8 relative to Case 9. We note that the cost savings will be examined under the aforementioned assumption of no arbitrage (i.e., $P - d > S$).

Case 1): $q > D > 0$, $(q - D) = R(t_e - t_b)$ and $Q_s > 0$ at t_e.

In this case, the optimal disposal quantity D_1^* is uniquely determined by the constraint $(q - D) = R(t_e - t_b)$. Hence, it can be easily verified that $D_1^* = q - R(t_e - t_b)$ and $Q_{s1}^* = Q_z$.

Case 2): $q > D > 0$, $(q - D) > R(t_e - t_b)$ and $Q_s > 0$ at t_e.

When $(q - D) > R(t_e - t_b)$ and $Q_s > 0$ at t_e, the next regular EOQ replenishment occurs $(q - D + Q_s)/R$ time units after t_b. In order to measure the cost saving of Case 2 over Case 9, the total costs of Case 2 and Case 9 will be calculated for the time duration of $(q - D + Q_s)/R$ (see e.g., Tersine [1]). The total cost for the duration of $(q - D + Q_s)/R$ for Case 2, TC_2, can be expressed as follows.

$$TC_2 = K - DS + \frac{(q-D)^2 PF}{2R} + \left(\frac{q-D-R(t_e-t_b)}{R}\right) Q_s (P-d)F + \frac{Q_s^2 (P-d)F}{2R} + C + (P-d)Q_s \tag{11}$$

The total cost for the same duration for Case 9, TC_{92}, is given by

$$TC_{92} = \frac{q^2 PF}{2R} + \frac{Q_s - D}{R}(PR + (2CRPF)^{0.5}) \tag{12}$$

From the relation (11) and (12), the cost saving of Case 2 over Case 9, CS_2 is given by $CS_2 = TC_{92} - TC_2$. The objective now is to find the optimal disposal amount D_2^* (which will maximize the cost saving CS_2), Namely, we will solve the following problem for D_2^*.

$$\underset{D}{\text{Maximize}}\ CS_2 = TC_{92} - TC_2 \tag{13}$$

From (13), it can be easily verified that

$$D_2^* = q + \frac{(P-d)R(t_e - t_b)}{d} - \frac{(P-d-S)R}{dF} \tag{14}$$

$$Q_{s2}^* = Q_z - [q - D_2^* - R(t_e - t_b)] \tag{15}$$

By examining Case 3 through Case 8 in a similar way, we have the following.

Case 3): $q > D > 0$, $(q - D) > R(t_e - t_b)$ and $Q_s = 0$ at t_e.

$$D_3^* = q - \frac{(P-S)R}{PF} - Q_o \tag{16}$$

$$Q_{s3}^* = 0. \tag{17}$$

Case 4): $q > D > 0$, $(q - D) < R(t_e - t_b)$ and $Q_s > 0$ at t_o.

$$D_4^* = q - \frac{(P-S)R}{PF} - Q_o \tag{18}$$

$$Q_{s4}^* = Q_z. \tag{19}$$

Case 5): $q > D > 0$, $(q - D) < R(t_e - t_b)$, $Q_s^1 = R(t_e - t_o)$ at t_o, and $Q_s > 0$ at t_e.

$$D_5^* = q - \frac{[(P-d)F(t_e - t_b) + (P-d) - S]R}{F(2P-d)} \tag{20}$$

$$Q_{s5}^* = Q_z. \tag{21}$$

Case 6): $D = q$, $Q_s > 0$ at t_b.

$$D_6^* = q \tag{22}$$

$$Q_{s6}^* = Q_z. \tag{23}$$

Case 7): $D = q$, $Q_s^1 = R(t_e - t_b)$ at t_b, and $Q_s > 0$ at t_e.

$$D_7^* = q \tag{24}$$

$$Q_{s7}^* = Q_z. \tag{25}$$

Case 8): $D = 0$, $Q_s > 0$ at t_e.

$$D_8^* = 0 \tag{26}$$

$$Q_{s8}^* = Q_z - [q - R(t_e - t_b)] \tag{27}$$

From equations (16) and (18), we note that the optimal disposal quantities D_3^* and D_4^* are strictly less than zero (i.e., $q - \frac{(P-S)R}{PF} - Q_0 < 0$). It is unrealistic for the disposal quantity to be negative. Hence, Case 3 or Case 4 will never be an optimal policy and they can be eliminated from any further consideration. Next, by directly comparing the optimal savings, CS_5^*, CS_6^* and CS_7^* relative to CS_1^*, it can be easily verified that $CS_1^* - CS_5^* > 0$, $CS_1^* - CS_6^* > 0$ and $CS_1^* - CS_7^* > 0$. It indicates that the optimal decisions of Case 1 dominate the optimal decisions of Case 5, Case 6 and Case 7. Therefore, Case 5, Case 6 or Case 7 will never be an optimal policy.

So far, we have excluded the possibilities of an optimal policy existing for Cases 3, 4, 5, 6, and 7. Therefore, the possible optimal policies can be listed as follows.

Case 1): $q > D > 0$, $(q - D) = R(t_e - t_b)$ and $Q_s > 0$ at t_e.

Case 2): $q > D > 0$, $(q - D) > R(t_e - t_b)$ and $Q_s > 0$ at t_e.

Case 8): $D = 0$, $Q_s > 0$ at t_e.

Case 9): $D = 0$ and $Q_s = 0$ (i.e. "Non-Response" policy).

We note that Policy 1 may be the optimal policy only if $CS_1^* > 0$. The corresponding conditions under which Case 1, 2, 8, and 9 may be the optimal policy are summarized in Table 1. Also, we note that the conditions in Table 1 are necessary conditions for the optimal policy. If there are more than one case with the necessary conditions satisfied, then the optimal cost saving of each case will be computed and the case with the maximum optimal cost saving will be the optimal policy.

Table 1. The conditions under which case 1, 2, 8, or 9 may be optimal

Case	Conditions
1	$CS_1^* > 0$
2	$q > D_2^* > 0$, $Q_{s2}^* > 0$, $q - D_2^* > R(t_e - t_b)$, $CS_2^* > 0$
8	$CS_8^* > 0$
9	the conditions for case 1, 2, abd 8 do not hold

4. ONE REGULAR EOQ REPLENISHMENT POINT DURING THE SALE ($q < R(t_e-t_b)$)

4.1 Description of Exclusive and Exhaustive Cases

In this section, we consider the case that there exists one regular EOQ replenishment point during the sale (i.e., $q < R(t_e-t_b)$). We note that Propositions 1, 2, and 3 still hold for the case of $q < R(t_e-t_b)$. Therefore, the feasible policies can be classified into the following seven mutually exclusive and exhaustive cases.

Case A): $0 < D < q$ and $Q_s > 0$ at t_0.

Case B): $0 < D < q$, $Q_s^1 = R(t_e-t_0)$ at t_0, and $Q_s > 0$ at t_e.

Case C): $D = q$, $Q_s > 0$ at t_b.

Case D): $D = q$, $Q_s^1 = R(t_e-t_b)$ at t_b, and $Q_s > 0$ at t_e.

Case E): $D = 0$, $Q_s > 0$ at t_0.

Case F): $D = 0$, $Q_s^1 = R(t_e - t_o)$ at t_o, and $Q_s > 0$ at t_e.

Case G): $D = 0$ and $Q_s = 0$ (i.e. "Non-Response" policy).

Given the above seven cases, we will employ Case G of no-special-order and no-disposal (i.e., wait until the remnant inventory is depleted and then purchase $Q_o = (\frac{2CR}{PF})^{0.5}$ for all subsequent orders) as the benchmark (see e.g., Tersine [1]). Cases A through F will be examined against this benchmark to determine the optimal disposal amount at time point t_b.

4.2 Cost Saving Comparisons for the case $q < R(t_e - t_b)$

In this section, we will examine the cost savings of Case A through Case F relative to Case G. We note that the cost savings will be examined under the aforementioned assumption of no arbitrage (i.e., $P - d > S$). By performing similar formulations and manipulations discussed for the case $q \geq R(t_e - t_b)$ in Section 3.2, we can have the following results for cost saving comparisons.

Case A): $0 < D < q$ and $Q_s > 0$ at t_o.

$$D_A^* = q - \frac{(P-S)R}{PF} - Q_o \tag{28}$$

$$Q_{sA}^* = Q_z \tag{29}$$

Case B): $0 < D < q$, $Q_s^1 = R(t_e - t_o)$ at t_o, and $Q_s > 0$ at t_e.

$$D_B^* = q - \frac{(P-d)R(t_e - t_b)}{2P-d} - \frac{R(P-d-S)}{F(2P-d)} \tag{30}$$

$$Q_{sB}^* = Q_z \tag{31}$$

Case C): $D = q$, $Q_s > 0$ at t_b.

$$D_C^* = q \tag{32}$$

$$Q_{sC}^* = Q_z \tag{33}$$

Case D): $D = q$, $Q_s^1 = R(t_e - t_b)$ at t_b, and $Q_s > 0$ at t_e.

$$D_D^* = q \tag{34}$$

$$Q_{sD}^* = Q_z \tag{35}$$

Case E): $D = 0$, $Q_s > 0$ at t_o.

$$D_E^* = 0 \tag{36}$$

$$Q_{sE}^* = Q_z \tag{37}$$

Case F): $D = 0$, $Q_s^1 = R(t_e - t_o)$ at t_o, and $Q_s > 0$ at t_e.

$$D_F^* = 0 \tag{38}$$

$$Q_{sF}^* = Q_z \tag{39}$$

From equation (28), we note that the optimal disposal quantity D_A^* is strictly less than zero (i.e., $q - \frac{(P-S)R}{PF} - Q_o < 0$). It is unrealistic for the disposal quantity to be negative. Hence, Case A will never be an optimal policy and it can be eliminated from any further consideration. Next, by directly comparing the optimal savings, CS_C^* relative to CS_E^* and CS_D^* relative to CS_B^*, it can be easily verified that $CS_E^* - CS_C^* > 0$ and $CS_B^* - CS_D^* > 0$. It indicates that the optimal decisions of Case E dominate the optimal decisions of Case C and the optimal decisions of Case B dominate the optimal decisions of Case D. Therefore, Case C or Case D will never be an optimal policy. Also, we note that CS_E^* is strictly greater than zero. Hence, Case G (i.e., "Non-Response" policy) will never be an optimal policy.

So far, we have excluded the possibilities of Cases A, C, D, and G being an optimal policy. Therefore, the possible optimal policies can be listed as follows.

Case B): $0 < D < q$, $Q_s^1 = R(t_e - t_o)$ at t_o, and $Q_s > 0$ at t_e.

Case E): $D = 0$, $Q_s > 0$ at t_o.

Case F): $D = 0$, $Q_s^1 = R(t_e - t_o)$ at t_o, and $Q_s > 0$ at t_e.

The corresponding conditions under which Policies B, E, and F may be the optimal policy are summarized in Table 2.

Table 2. The conditions under which case B, E, or F may be optimal

Case	Conditions
B	$q > D_B^* > 0$, $CS_B^* > 0$
E	
F	$CS_F^* > 0$

As in section 3.2, we note that the conditions shown in Table 2 are necessary conditions for the optimal policy. If there are more than one case with the necessary conditions satisfied, then the optimal cost saving of each case will be computed and the case with the maximum optimal cost saving will be the optimal policy.

5. DECISION PROCESS AND NUMERICAL RESULTS

Thus far, we have formulated the mathematical model and derived the optimal inventory and disposal policies. In this section, we first elaborate on the decision process that effectively leads to the optimal policy. Next, under given sets of parameter values, we demonstrate that with small variations in parameter values, all seven cases (four under $q \geq R(t_e - t_b)$ and three under $q < R(t_e - t_b)$) will become optimal policies. We also provide additional managerial insights.

5.1 Decision Process

By comparing the possible optimal policies provided in subsections 3.2 and 4.2, we have the following results.

1. Considering Case 8 and Case 9 in subsection 3.2, if $(\frac{P-d}{P})^{0.5}(\frac{Q_{s8}}{Q_0}) > 1$, then Case 9 dominates Case 8. Otherwise, Case 8 dominates Case 9.
2. Considering Case E and Case F in subsection 4.2, if $d < a$, then Case E dominates Case F, where

$$a = P - (\frac{PR+(2CRPF)^{0.5}}{R} - \frac{C}{R(t_e - t_b) - q})(\frac{2}{2+F(t_e - t_b - (q/R))}).$$

   Otherwise, Case F dominates Case E.

By incorporating the results described above, the decision process can be simplified as the diagram shown in Figure 3.

5.2 Numerical Results

In this subsection, we demonstrate that all seven cases can be optimal policies with only one or two changes in values of parameters. To achieve our objective, we select the discount magnitude d and the on-hand inventory level q at t_b as the parameters whose values change. Example 1 is designed to study the cases of $q \geq R(t_e - t_b)$ while Example 2 is to study the cases of $q < R(t_e - t_b)$. The following values of the parameters are employed for both Example 1 and Example 2: P=100, R=800, K=0, S=18, C=5490, and F=0.5.

__Example 1.__ Given the sale period $t_e - t_b$=0.3, we perform the sensitivity analysis on optimal inventory and disposal polices with respect to on-hand

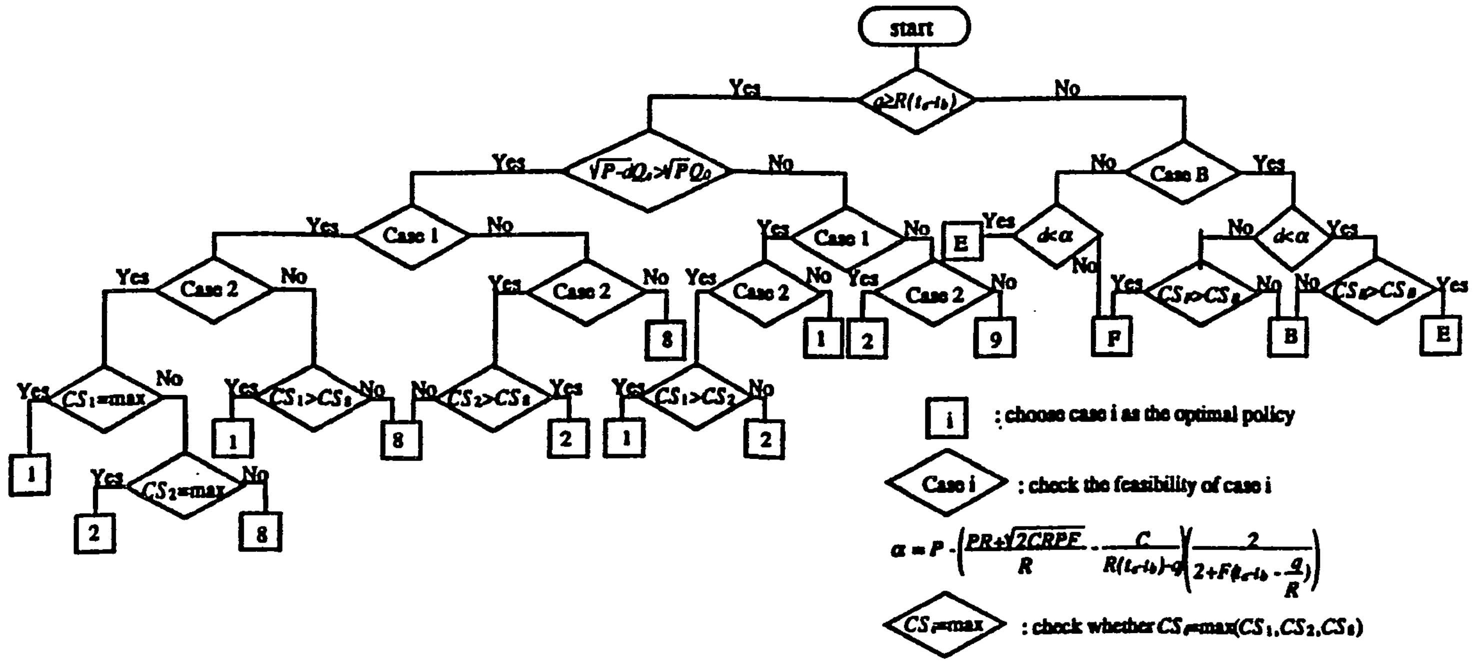

Figure 3. The decision process for the optimal inventory and disposal policies

inventory q and discount magnitude d. Table 3 presents the results. By examining Table 3 carefully, we make the following interesting observations for Example 1.

1) When the discount magnitude d is sufficiently small and the on-hand inventory level q at t_b is large enough (e.g., $d = 1$ and $q \geq 275$), the policy of Case 9 is optimal (i.e., "Non-Response" policy).
2) When the discount magnitude d is neither too large nor too small (e.g., $d = 20$, 35 or 50), the policy of Case 8 is optimal (i.e., do not dispose at t_b but place a special order Q^*_{s8} at t_e) regardless of the on-hand inventory level q at t_b.
3) When the discount magnitude d is reasonably large (e.g., $d = 65$) and the on-hand inventory level q at t_b is sufficiently high (e.g., $q \geq 310$), the policy of Case 2 is optimal (i.e., dispose D^*_2 at t_b and place a special order Q^*_{s2} at t_e).
4) When the discount magnitude d is sufficiently large (e.g., $d = 80$), the policy of Case 1 is optimal (i.e., dispose D^*_1 at t_b and place a special order Q^*_{s1} at t_e) regardless of the on-hand inventory level q at t_b.

<u>Example 2.</u> All the parameter values are the same as Example 1 with the exception that the sale period $t_e - t_b = 0.5$. The corresponding results are shown in Table 4.

By examining Table 4 carefully, we make the following interesting observations for Example 2.

Table 3. Optimal inventory and disposal policies for $q \geq \ell(t_e - t_b)$

	$d=1$		$d=5$		$d=20$		$d=35$		$d=50$		$d=65$		$d=80$	
$q=245$	8	357	8	2550	8	15621	8	40297	8	87024	8	184152	1	445152
$q=275$	9	0	8	1650	8	14265	8	38485	8	84756	8	181429	1	442393
$q=310$	9	0	8	667	8	12740	8	36418	8	82147	2	178286	1	439246
$q=345$	9	0	9	0	8	11276	8	34401	8	79576	2	175216	1	436176
$q=380$	9	0	9	0	8	9874	8	32433	8	77043	2	172222	1	433182
$q=415$	9	0	9	0	8	8532	8	30515	8	74548	2	169305	1	430264

Table 4. Optimal inventory and disposal policies for $q < \ell(t_e - t_b)$

	$d=1$		$d=5$		$d=20$		$d=35$		$d=50$		$d=65$		$d=80$	
$q=20$	F	390	F	2413	F	14825	F	38842	F	84910	F	181380	F	441656
$q=65$	F	162	F	1964	F	13550	F	36742	F	81948	F	177628	F	437078
$q=110$	E	13	F	1396	F	12175	F	34559	F	78995	F	173832	B	465084
$q=155$	E	13	F	706	F	10698	F	32295	F	75942	F	169992	B	427989
$q=200$	E	13	E	335	F	9119	F	29948	F	72826	F	166107	B	423619
$q=245$	E	13	E	335	F	7440	F	27518	F	69647	F	162178	B	419376
$q=245$	E	13	E	335	E	6370	F	25007	F	66405	F	158204	B	415260
$q=245$	E	13	E	335	E	6370	E	24011	E	63702	B	154224	B	411270
$q=245$	E	13	E	335	E	6370	E	24011	E	63702	E	153795	E	407693

1) When the discount magnitude d is sufficiently small (e.g., $d = 5$) and the on-hand inventory q at t_b is sufficiently high (e.g., $q \geq 200$), the policy of Case E is optimal (i.e., do not dispose at t_b but place a special order Q^*_{sE} at t_o).
2) When the on-hand inventory q at t_b is sufficiently small (e.g., $q =$ 20 or 65), the policy of Case F is optimal (i.e., do not dispose at t_b but place a order $Q^1_s = R(t_e - t_o)$ at t_o and place a special order Q^*_{sF} at t_e) regardless of the discount magnitude d.
3) When the discount magnitude d is sufficiently high (e.g., $d = 80$) and the on-hand inventory level q at t_b is neither too high nor too low (e.g., $110 \leq q \leq 335$), the policy of Case B is optimal (i.e., dispose D^*_B at t_b, place a special order $Q^1_s = R(t_e - t_o)$ at t_o, and place a special order Q^*_{sB} at t_e).

Furthermore, we note that the following properties can be easily verified by way of simple calculus.

<u>Property 1.</u> $\frac{\partial CS}{\partial q} \leq 0$ for all cases.

<u>Property 2.</u> $\frac{\partial CS}{\partial d} \geq 0$ and $\frac{\partial^2 CS}{\partial d^2} \geq 0$ for all cases.

Property 1 implies that when the on-hand inventory q increases, the optimal cost saving for the inventory and disposal policies will decrease or remain the same.

Meanwhile, Property 2 implies when the discount magnitude d increases, the optimal cost saving for the inventory and disposal policies will increase or remain the same. In addition, the difference in the increase of the optimal cost saving increases as the discount magnitude d increases.

6. CONCLUSIONS

In this paper, we constructed and analyzed an EOQ-type model for a buyer who was just informed of a temporary sale. For such a buyer, optimal inventory and disposal policies were derived by comparing cost savings of various cases. By analyzing the optimal inventory and disposal policies, several managerial insights were obtained. In particular, as the discount magnitude d increases, the optimal cost saving will increase or remain the same. On the other hand, as the on-hand inventory level q at t_b increases, the optimal cost saving will decrease or remain the same (this is consistent with Theorem 1 in Ardalan [3]).

This paper can be viewed as an exploratory investigation of integrating the inventory policies in response to sales and the inventory policies with disposal options. Therefore, numerous extensions that will enhance the model presented in this paper can be made. For examples, one class of extensions can be made with respect to the duration of the sale. That is, the duration of a sale may be relatively long (e.g., longer than one regular EOQ cycle).

Another class of extensions can be made with respect to the time at hich the sale is known to the buyer and to the time at which the sale is in effect. An additional class of extensions can be made with respect to policies of a seller regarding buyers' disposals. Implicitly, in this paper, it is assumed that the seller will not react to the buyers' disposal (if any). It would be of interest to investigate several possible policies of a seller. e.g., prohibition of disposals, benefit sharing of disposals, etc. We believe that such extensions will improve the applicability in practice of the inventory/disposal models in response to sales. We hope this improvement in applicability, in turn, will result in increased economic efficiency for the buyer (as well as the seller).

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CHAPTER II.

OPTIMAL INVENTORY POLICIES IN RESPONSE TO A PRE-ANNOUNCED SALE

A paper to be submitted to *IIE Transactions*

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ABSTRACT

We construct and analyze an EOQ-type model for a buyer who is just informed of a pre-announced sale. By "a pre-announced sale", we mean the announcement time of the sale occurs before the beginning time of the sale. Under the pre-announced sale, the buyer is assumed to have an option to adjust his replenishment strategy before the sale is effective and an option to place special orders during the temporary sale. For such a buyer, optimal inventory policies are derived by comparing cost savings of various cases. By analyzing the optimal inventory policies, several managerial insights are obtained. For example, as the period between the announcement of the sale and the commencement of the sale increases, the optimal cost saving will increase or remain the same. In addition, as the duration of the sale increases, the optimal cost saving will increase or remain the same.

1. Introduction

In this paper, an EOQ-type model is constructed and analyzed for a buyer who is just informed of an announcement from his supplier that there will be a temporary sale in the near future. Under the pre-announced sale, the buyer is assumed to have an option to adjust his replenishment strategy before the temporary sale is effective and an option to place special orders during the temporary sale. By comparing cost savings of various cases of strategies (see e.g., Tersine, 1994), we obtain the optimal solutions for the inventory replenishment strategies. By analyzing the optimal solutions, we obtain interesting managerial insights for the buyer.

The optimal inventory policies under price changes (increases or decreases), based on the classical economic order quantity (EOQ) models, have been extensively studied (see e.g., Taylor and Bradely, 1985; Lev and Weiss, 1990). Also, for temporary price decreases, there have been numerous studies investigating the optimal replenishment and inventory policies (see e.g., Ardalan, 1988, 1994 or Aull-Hyde, 1992). Ardalan (1994) deals with a temporary price discount and derives the optimal inventory policies by employing a net present value method and/or by incorporating the marketing effect on demand. Aull-Hyde (1992) discusses the optimal ordering rules in response to supplier restrictions on special order sizes that accompany temporary price decreases. In Tersine and Barman (1995), a composite EOQ model, which can be disaggregated into several traditional EOQ models, is developed to determine the optimal levels of order quantity and backorder quantity in response to a temporary price discount. We note that the models constructed and analyzed in the last three papers assume that the sale period is short relative to the regular EOQ replenishment cycle and the sale period is within a regular EOQ replenishment cycle. Also, there exists an implicit assumption in the last three papers that the announcement time of the temporary sale is identical to the beginning time of the sale.

The numerous studies of the topic of the inventory policies with temporary price discounts in the literature reflect the importance of the topic to both academicians and practitioners. Also, it is intuitive that, given a pre-announced temporary sale, a buyer may find it beneficial to adjust his replenishment strategy before the temporary sale and/or place special orders during the sale at a reduced price because these transactions may result in reduced total cost for the inventory system. Up until now, however, there have been few analytical models that investigate the inventory replenishment policies under pre-announced temporary sale. Hence, considering the fact that numerous firms utilize EOQ-based decision making processes for such policies, it is highly desirable to construct and analyze EOQ-based models of inventory policies under pre-announced temporary sale.

In this paper, we will focus on optimal inventory replenishment policies for a buyer who is just informed of an announcement from his supplier that there will be a temporary sale in the near future. By "just informed," we mean that the buyer is able to respond to the temporary sale from that time point on. That is, the emphasis is on **when** the buyer is **able to respond** to a sale. Hence, if the buyer is able to respond to a sale from a particular time point on due to administrative, informational, organizational, and/or other reasons, that particular time point is viewed as the time point at which the buyer is "just informed". Also, we want to point out that the implicit assumption that the announcement time of the temporary sale is identical to the beginning time of the temporary sale in previous publications (see e.g., Ardalan, 1988, 1994; Aull-Hyde, 1992; Tersine and Barman, 1995) is relaxed in this paper. In contrast to the previous literatures, we assume that the announcement time of the temporary sale occurs earlier than the beginning time of the temporary sale. This is what we mean by "pre-announced". In addition, by "temporary sale," we mean that the sale period is short relative to the regular EOQ replenishment cycle. Specifically, we will restrict our attention to the case that the sale period is less than one regular EOQ replenishment cycle. We note that the sale period could actually be quite long in absolute duration (e.g., 3 months) when the regular EOQ replenishment cycle is also long in absolute duration (e.g., 6 months). Hence, this assumption is not as restrictive as it may first appear and such an assumption can be found in several publications (see e.g., Ardalan, 1988, 1994; Aull-Hyde, 1992; Tersine and Barman, 1995).

The rest of this paper is organized as follows. We first introduce the model environments and the possible sets for the pre-announced temporary sale. Next, we obtain the optimal solutions by comparing the cost saving of various cases. We then present the decision process for the optimal inventory replenishment policies and provide illustrative numerical examples. From the numerical results, several managerial insights and properties are derived. Finally, we summarize and comment on further research.

2. Model Environments: Assumptions and Definitions

In our model, a buyer determines the optimal order quantity from his supplier based on the classical EOQ model. As in numerous EOQ-type models, we make the following assumptions.

1) the buyer's demand is constant over time,
2) no shortage is allowed,
3) replenishment is instantaneous,
4) lead time is zero.

We note that, the assumption of zero lead time is made for simplicity and a positive lead time can be easily incorporated into our model. Also, the following definitions of the classical EOQ model are employed.

R: the buyer's demand per unit time (e.g., annual demand).

P: the purchase price per unit to the buyer from the supplier before and after the temporary sale.

F: the holding cost per unit time as a fraction of the unit purchase price.

C: the ordering (setup) cost per order (i.e., a fixed cost independent of the order quantity).

Q_0: the economic order quantity given the purchasing price per unit, P.

i.e., $Q_0 = (\frac{2CR}{PF})^{0.5}$.

We also note that the inventory holding cost per unit time F is assumed to be a fraction of the original unit purchase price.

Let us suppose that the buyer, at time point t_a, is informed that there is a sale effective from t_b through t_e, and the buyer is expected to make his decisions regarding his inventory policies. As mentioned earlier, we also assume that the sale period is less than one regular EOQ replenishment cycle (i.e., $t_e - t_b < (\frac{2C}{PFR})^{0.5}$). Also, we note that the relationship of $t_a < t_b < t_e$ holds throughout the rest of this paper.

We will denote the magnitude of price decrease in the sale by d ($d > 0$), and the new purchasing price per unit for the buyer will be $P - d$ during the sale. Also, we denote the on-hand inventory level (stock position) at time point t_a by q and the q units of inventory will be depleted at time point t_o (i.e., $t_o = t_a + \frac{q}{R}$). Let us assume that the buyer has the option to respond to the pre-announced temporary sale after the buyer is informed of the sale at t_a. Given this option, the buyer must determine the optimal inventory policies from t_a to t_e. In response to the pre-announced temporary sale, adjusting the inventory replenishment strategy from t_a *to* t_b and/or placing special order(s) at the decreased price ($P - d$) during the sale can be beneficial to the buyer because these transactions may result in reduced total cost of the inventory system. By examining the time sequences of t_a, t_b, t_e, and t_o, we can have the following three mutually exclusive and exhaustive sets of precedence relationships under the assumption that the sale period is less than one regular EOQ replenishment cycle.

Set A: $t_a < t_b < t_e \leq t_o$.

Set B: $t_a < t_b \leq t_o < t_e$.

Set C: $t_a \leq t_o < t_b < t_e$.

For our models, we will optimally determine the special order quantities and the time points at which special orders are placed for the above three sets. The following three sections will investigate these three sets and derive the corresponding optimal solutions for the inventory policies. Also, we note that the earliest time for the buyer to respond to the announced temporary sale is at t_b for Set A while they are at t_o for Set B and Set C. Finally, throughout the rest of the paper, we will assume that the products are withdrawn from inventory on a first-in, first-out (FIFO) basis. This is a reasonable assumption in numerous practical inventory systems, and it facilitates tractable construction and analysis of the model.

3. Set A: $t_a < t_b < t_e \leq t_o$

In this section, we consider the set that t_a, t_b, and t_e are all within an regular EOQ replenishment cycle (i.e., $t_a < t_b < t_e \leq t_o$). Figure 1 illustrates two possible policies for the buyer to follow. One is to place a special order during the sale. We will call this policy the "Response" policy. The other one is to ignore the option to place a special order during the sale. We will call this policy the "Non-Response" policy. The following Lemma determines the optimal time point at which the special order is placed for the "Respond" policy.

<u>**Lemma 1.**</u> The optimal time point at which the special order is placed for the set A: $t_a < t_b < t_e \leq t_o$ is at t_e.

Proof: See Appendix.

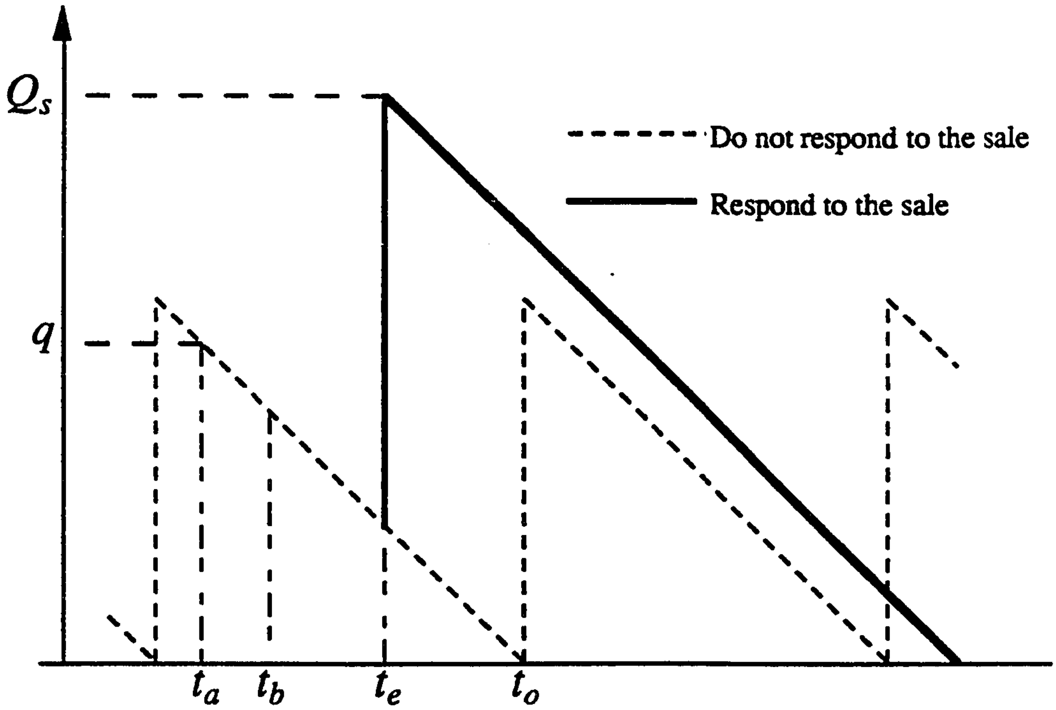


Figure 1. The case for $t_a < t_b < t_e \leq t_o$

We will denote the on-hand inventory level at t_e by $\bar{q}$ (i.e., $\bar{q} = q - R(t_e - t_a)$). Also, we will denote the inventory level at t_e after the special order (including the inventory before the special order, i.e., $\bar{q}$) by Q_s. In order to measure the cost saving of the "Response" policy over the "Non-Response" policy accurately, the total cost will be calculated from t_e to $(t_e + \frac{Q_s}{R})$ (see e.g., Tersine, 1994). The total cost from the time point t_e to the time point $(t_e + \frac{Q_s}{R})$ for the "Response" policy, TC_R, can be expressed as follows.

$$TC_R = \frac{\bar{q}^2 PF}{2R} + (Q_s - \bar{q})(P-d) + (Q_s - \bar{q})(\frac{\bar{q}}{R})(P-d)F + \frac{(Q_s - \bar{q})^2 (P-d)F}{2R} + C \tag{1}$$

The corresponding total cost for the same duration for the "Non-Response" policy, TC_{NR}, is given by

$$TC_{NR} = \frac{\bar{q}^2 PF}{2R} + \frac{Q_s - \bar{q}}{R} (PR + \sqrt{2CRPF}) \tag{2}$$

From the equations (1) and (2), the cost saving of the "Response" policy over the "Non-Response" policy is given by $G_1 = TC_{NR} - TC_R$. The objective now is to find the optimal Q_s which will maximize G_1. Namely,

$$\underset{Q_s}{\text{Maximize}}\; G_1 = TC_{NR} - TC_R \tag{3}$$

From the maximization of the above problem, the following first derivatives can be easily obtained.

$$\frac{\partial G_1}{\partial Q_s} = \frac{PR + \sqrt{2CRPF}}{R} - (P-d) - \frac{\bar{q}(P-d)F}{R} - \frac{(Q_s - \bar{q})(P-d)F}{R} \tag{4}$$

By setting equation (4) equal to zero, the optimal Q^*_{sG1} can be obtained as follows

$$Q^*_{sG1} = \frac{dR + \sqrt{2CRPF}}{(P - d)F} = \frac{PR + \sqrt{2CRPF}}{(P - d)F} - \frac{R}{F} = \frac{Rd}{(P-d)F} + \frac{P}{P-d}Q_o \qquad (5)$$

Equation (5) implies that if the buyer places a special order during the sale, the optimal strategy is to replenish the inventory up to the level $Q_s = \frac{dR + \sqrt{2CRPF}}{(P - d)F}$ at t_e regardless of the level of on-hand inventory at t_e. We note that the expression of Q^*_{sG1} in equation (5) is identical to the special order quantity shown in Tersine (1994) or Ardalan (1988) when on-hand inventory level is zero. Also, we will denote the quantity $\frac{dR + \sqrt{2CRPF}}{(P - d)F}$ by Q^*_s throughout the rest of this paper.

By substituting the closed-form solution of Q^*_{sG1} into (3) and by performing some algebraic manipulations, we can obtain the optimal cost saving G^*_1 as follows.

$$G^*_1 = C\ [(\frac{Q^*_s - \bar{q}}{Q_o(\frac{P}{P-d})^{0.5}})^2 - 1] \qquad (6)$$

In such a case, it is not always advantageous to place the special order during the sale. By examining the expression of G^*_1, we can have the following decision-making rules for the set that t_a, t_b, t_e, and t_o are all within a regular EOQ replenishment cycle.

If $(Q^*_s - \bar{q}) > Q_o(\frac{P}{P - d})^{0.5}$, then place a special order up to the inventory level Q^*_s at t_e.

Else ignore the announced temporary sale.

Exhibit 1. The decision-making rule for the set of $t_a < t_b < t_e \le t_o$.

We note that the result of Exhibit 1 is identical to the result in Tersine (1994) or Ardalan (1988) since t_a, t_b, and t_e are all within a regular EOQ cycle. We also note that, throughout the rest of the paper, we will employ the "Non-Response" policy as the benchmark and formulate the cost savings as the performance criteria.

4. Set B: $t_a < t_b \leq t_o < t_e$

In this section, we consider the set that t_a and t_b are within an EOQ replenishment cycle while t_e is within the EOQ replenishment cycle which follows the cycle contains t_a and t_b (i.e., $t_a < t_b \leq t_o < t_e$). We note that the buyer can either place a special order right at t_o or place a special order to meet the exact demand from t_o to t_e and an additional special order at t_e. Figure 2 illustrates these two "Respond" policies as well as the "Non-Response" policy for the buyer to follow. We will first examine the policy that places a special order at t_o. By performing similar formulations and manipulations discussed for the set $t_a < t_b < t_e \leq t_o$ and by considering the duration from t_o to $(t_o + \frac{Q_s}{R})$, we can easily obtain the cost saving G_2 of the policy that places a special order at t_o over "Non-Response" policy as follows.

$$\underset{Q_s}{\text{Maximize}}\ G_2 = \frac{Q_o}{R}[(P-d)R + \sqrt{2CR(P-d)F}\,] + (\frac{Q_s - Q_o}{R})(PR + \sqrt{2CRPF}\,)$$

$$- C - Q_s(P - d) - \frac{Q_s^2(P - d)F}{2R} \qquad (7)$$

From the maximization of the above problem, the following first derivative with respect to Q_s can be easily obtained.

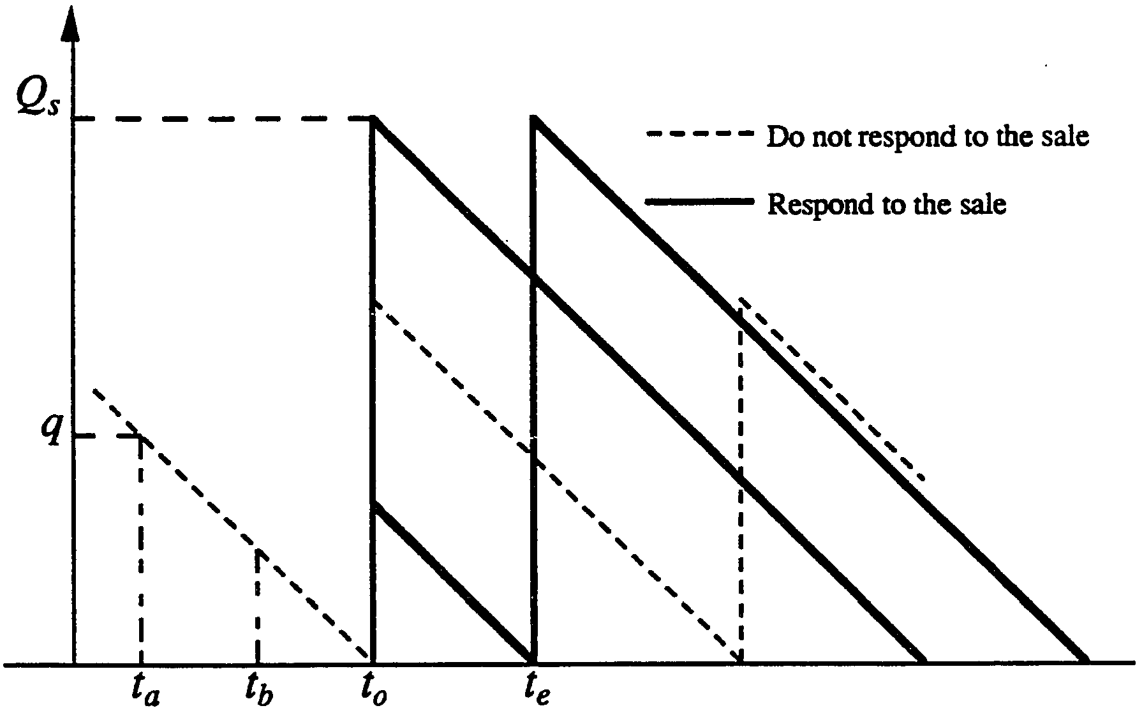


Figure 2. The case for $t_a < t_b \leq t_o < t_e$

$$\frac{\partial G_2}{\partial Q_s} = \frac{PR + \sqrt{2CRPF}}{R} - (P - d) - \frac{(P - d)FQ_s}{R} \tag{8}$$

By setting equation (8) to zero, the optimal Q^*_{sG2} can be obtained as follows.

$$Q^*_{sG2}\ (= Q^*_s) = \frac{dR + \sqrt{2CRPF}}{(P - d)F} = \frac{PR + \sqrt{2CRPF}}{(P - d)F} - \frac{R}{F} = \frac{Rd}{(P-d)F} + \frac{P}{P-d}Q_o \tag{9}$$

We note that the expression of Q^*_{sG2} in equation (9) is identical to the special order quantity shown in equation (5). By substituting the closed-form solution of Q^*_{sG2} into G_2 and by performing some algebraic manipulations, we can obtain the cost saving as follows.

$$G_2^* = \frac{C(P - d)}{P}\left(\frac{Q^*_s}{Q_o} - 1\right)^2 \tag{10}$$

From equation (10), we can easily concluded that, it is always advantageous to place the special order Q^*_s at t_o during the sale for the set of $t_a < t_b \le t_o < t_e$.

Now we proceed to examine the policy that places a special order to meet the exact demand from t_o to t_e and an additional special order at t_e. By considering the duration from t_o to $(t_e + \frac{Q_s}{R})$, we can easily obtain the cost saving G_3 of the policy that places a special order to meet the exact demand from t_o to t_e and an additional special order at t_e over the "Non-Response" policy as follows.

$$\begin{aligned}\underset{Q_s}{\text{Maximize}}\ G_3 = {} & \frac{Q_o}{R}[(P-d)R + \sqrt{2CR(P-d)F}] + [(t_e - t_o) + \frac{Q_s}{R} - \\ & \frac{Q_o}{R}](PR + \sqrt{2CRPF}) - C - Q_s(P - d) - \frac{Q_s^2(P - d)F}{2R} \\ & - C - R(t_e - t_o)(P - d) - \frac{R(t_e - t_o)^2(P - d)F}{2}\end{aligned} \tag{11}$$

From the maximization of the above problem, the following first derivative with respect to Q_s can be easily obtained.

$$\frac{\partial G_3}{\partial Q_s} = \frac{PR + \sqrt{2CRPF}}{R} - (P - d) - \frac{(P - d)FQ_s}{R} \qquad (12)$$

By setting equation (12) to zero, the optimal Q_s can be obtained as follows.

$$Q^*_{sG3}\ (= Q^*_s) = \frac{dR + \sqrt{2CRPF}}{(P - d)F} = \frac{PR+\sqrt{2CRPF}}{(P-d)F} - \frac{R}{F} = \frac{Rd}{(P-d)F} + \frac{P}{P-d}Q_o \qquad (13)$$

We note that the expression of Q^*_{sG3} in equation (13) is identical to the special order quantity shown in equation (5). The following proposition determines the optimal inventory strategies between the policy of placing a special order right at t_o and the policy of placing a special order to meet the exact demand from t_o to t_e and an additional special order at t_e.

__Proposition 1.__ Assume that, t_o occurs during the sale. Also, we denote the EOQ at price $(P - d)$ *by* $\hat{Q}$ (i.e., $\hat{Q} = (\frac{2CR}{(P - d)F})^{0.5}$).

If $(t_e - t_o) < \dfrac{Q_s - (Q_s^{*2} - \hat{Q}^2)^{0.5}}{R}$, then place a special order Q^*_s at t_o.

Else, place a special order to meet the exact demand from t_o to t_e and an additional special order Q^*_s at t_e.

__Proof:__

By comparing cost savings of the policy that places a special order Q^*_s right at t_o and the policy of placing a special order to meet the exact demand from t_o to t_e and an additional special order Q^*_s at t_e (i.e., G^*_2 vs G^*_3), we note that if $G^*_2 - G^*_3 > 0$, then a special order of Q^*_s at t_o will be the optimal policy. After some eliminations of identical terms, we can obtained the following relation for $G^*_2 - G^*_3$.

$$G_2^* - G_3^* = C + \frac{(t_e - t_o)^2 R(P-d)F}{2} + (t_e - t_o)R(P-d) - (t_e - t_o)(PR + \sqrt{2CRPF})$$

After some algebraic manipulations, we note that $G_2^* - G_3^* > 0$ if $(t_e - t_o) > \frac{Q_s + (Q_s^{*2} - \hat{Q}^2)^{0.5}}{R}$ or $(t_e - t_o) < \frac{Q_s - (Q_s^{*2} - \hat{Q}^2)^{0.5}}{R}$. We note that $(t_e - t_o) > \frac{Q_s + (Q_s^{*2} - \hat{Q}^2)^{0.5}}{R}$ will not be further considered since $(t_e - t_b)$ is less than one regular EOQ replenishment cycle. Therefore, the only condition that enables $G_2^* - G_3^* > 0$ is $(t_e - t_o) < \frac{Q_s - (Q_s^{*2} - \hat{Q}^2)^{0.5}}{R}$. We note that Proposition 1 is an extension of Corollary 1 in Ardalan (1988) which only considers the policy of placing a special order Q_s^* at t_o. We note that Proposition 1 also explicitly states the decision-making rule for the set of $t_a < t_b \le t_o < t_e$.

5. Set C: $t_a \le t_o < t_b < t_e$

In this section, we investigate the set that t_o $(= t_a + \frac{q}{R})$ comes before the beginning time of the sale t_b (i.e., $t_a \le t_o < t_b < t_e$). According to the Theorem 4 in Lev and Weiss (1990), we note that all of the orders placed from t_o to t_b (excluding the time point t_b) are of the same size. Furthermore, we assume that the inventory of the last order before t_b is depleted at time point t_x (i.e., $t_b \le t_x \le t_e$). Let us denote the integer number of the equal-size orders from t_o to t_x by n. We note that the possible optimal strategy for the buyer from t_x to t_e is either to place a special order Q_s at t_x or to place a special order to meet the exact demand from t_x to t_e and an additional special order Q_s at t_e. Figure 3 illustrates these two possible optimal strategies as well as

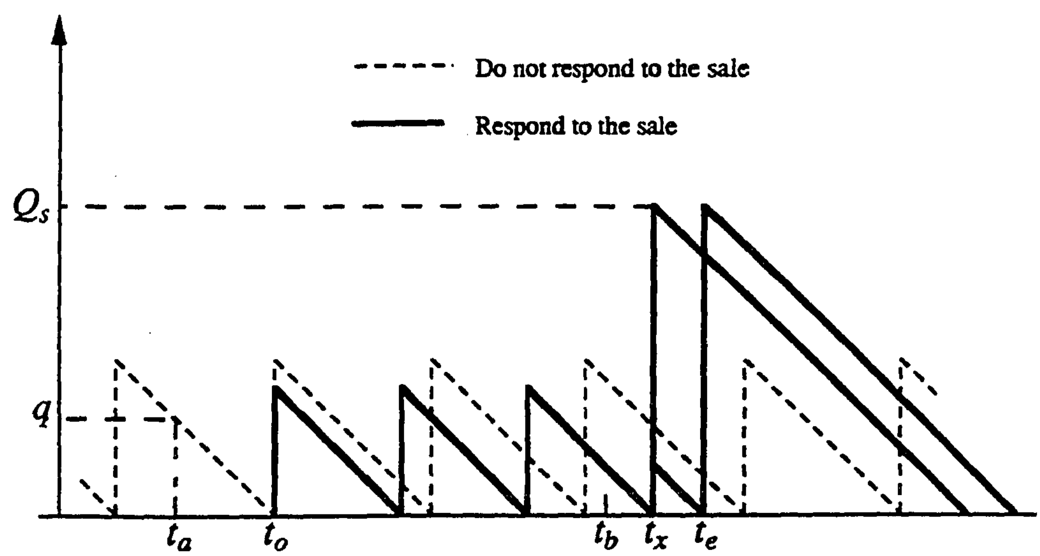


Figure 3. The case for $t_a \leq t_o < t_b < t_e$

the "Non-Response" policy for the buyer to follow. Throughout the rest of the paper, we will denote the strategy of placing a special order Q_s at t_x as "One Special Order Policy" while the strategy of placing a special order to meet the exact demand from t_x to t_e and an additional special order Q_s at t_e as "Two Special Orders Policy". The following two subsections will investigate these two possible optimal strategies.

5.1 One Special Order Policy

In this subsection, we investigate the strategy of placing n equal-size orders from t_o to t_x and a special order Q_s right at t_x. As in the preceding sections, we will employ the "Non-Response" policy as the benchmark. By considering the duration from t_o to $(t_x + \frac{Q_s}{R})$, we can easily obtain the cost saving of the "One Special Order" policy over the "Non-Response" policy as follows.

$$\underset{n, t_x, Q_s}{\text{Maximize}}\; G_4 = [\frac{Q_s}{R} + (t_x - t_o)](PR + \sqrt{2CRPF}) - nC - PR(t_x - t_o)$$
$$- \frac{RPF(t_x - t_o)^2}{2n} - C - (P - d)Q_s - \frac{Q_s^2(P - d)F}{2} \quad (14)$$

subject to: $t_b \leq t_x \leq t_e$

n is an integer

The above objective function G_4 is for the case where there is no regular EOQ replenishment point during the sale. If there is a regular EOQ replenishment point during the sale, the the objective function becomes as follows.

$$\underset{n,t_x,Q_s}{\text{Maximize}}\ G_5 = \left(\frac{Q_o}{R}\right)[(P-d)R + \sqrt{2CR(P-d)F}] + \left[\frac{Q_s-Q_o}{R} + (t_x-t_o)\right]$$

$$(PR + \sqrt{2CRPF}) - nC - PR(t_x-t_o) - \frac{RPF(t_x - t_o)^2}{2n}$$

$$- C - (P-d)Q_s - \frac{Q_s^2(P-d)F}{2} \qquad (15)$$

We note that the difference between objective functions G_4 and G_5 is constant. Therefore, the first derivatives of G_4 and G_5 are identical. From the maximization of the above problems, the following first derivative with respect to Q_s can be easily obtained.

$$\frac{\partial G_4}{\partial Q_s} = \frac{\partial G_5}{\partial Q_s} = \frac{PR + \sqrt{2CRPF}}{R} - (P - d) - \frac{(P - d)FQ_s}{R} \qquad (16)$$

By setting equation (16) to zero, the optimal Q_s can be obtained as follows.

$$Q^*_{sG4}\ (=Q^*_{sG4}=Q^*_s) = \frac{dR + \sqrt{2CRPF}}{(P - d)F} = \frac{PR + \sqrt{2CRPF}}{(P - d)F} - \frac{R}{F}$$
$$= \frac{Rd}{(P-d)F} + \frac{P}{P-d}Q_o \qquad (17)$$

We note that the expression of Q_s in equation (17) is identical to the special order quantity shown in equation (5). We also note that the special order quantity is independent from the other decision variables n and t_x.

Given t_x, the determination of the integer decision variable n can be treated as a finite horizon EOQ problem which is proposed and solved by Schwarz (1972). From Schwarz (1972), we note that the optimal solution for n, given t_x, is $n(t_x) = \lfloor 0.5 + (0.25 + (t_x - t_o)^2\frac{PRF}{2C})^{0.5}\rfloor$ where $\lfloor Y \rfloor$ is the largest integer less than or equal to Y. Throughout the rest of this paper, we will denote the optimal solution of n from t_o to t_x, t_o to t_b, and t_o to t_e by $n(t_x)$, $n(t_b)$, and $n(t_e)$, respectively. Moreover we note that $n(t_b) \leq n(t_x) \leq n(t_e)$ and $n(t_e)$ is equal to either $n(t_b)$ or $n(t_b) + 1$ (this is due to the assumption that the sale period is less than one regular EOQ replenishment cycle).

Given the integer variable n, we can have the first derivative with respect to t_x as follows.

$$\frac{\partial G_4}{\partial t_x} \left(= \frac{\partial G_5}{\partial t_x}\right) = (PR + \sqrt{2CRPF}) - \frac{(t_x - t_o)RPF}{n} - PR \qquad (18)$$

By rearranging equation (18), we can easily obtain the following relation.

$$Q_o = \left(\frac{2CR}{PF}\right)^{0.5} = \frac{R(t_x - t_o)}{n} \qquad (19)$$

The economic implication of relation (19) states that the optimal solution of t_x will be the regular EOQ replenishment point during the sale. Therefore, if there is a regular EOQ replenishment point during the sale, then the optimal t_x occurs at the regular EOQ replenishment point during the sale. On the other hand, if there is no regular EOQ replenishment point during the sale (i.e., t_b and t_e are within the same EOQ regular replenishment cycle), then the optimal solution of t_x occurs at t_b or t_e. This can be easily observed from Figure 4 where we plot cost saving as y axis and t_x as x axis under the integer constraint of the decision variable n. In Figure 4, the maximum cost saving occurs at the regular EOQ replenishment points. In order to determine the optimal solution of t_x, we define $\underline{n}$ by $\left\lfloor \frac{R(t_b - t_o)}{Q_o} \right\rfloor$ and $\bar{n} = \underline{n} + 1$. By observing Figure 4 carefully, we can conclude the decision-making rule shown in Exhibit 2 for the case that only one special order is allowed during the sale.

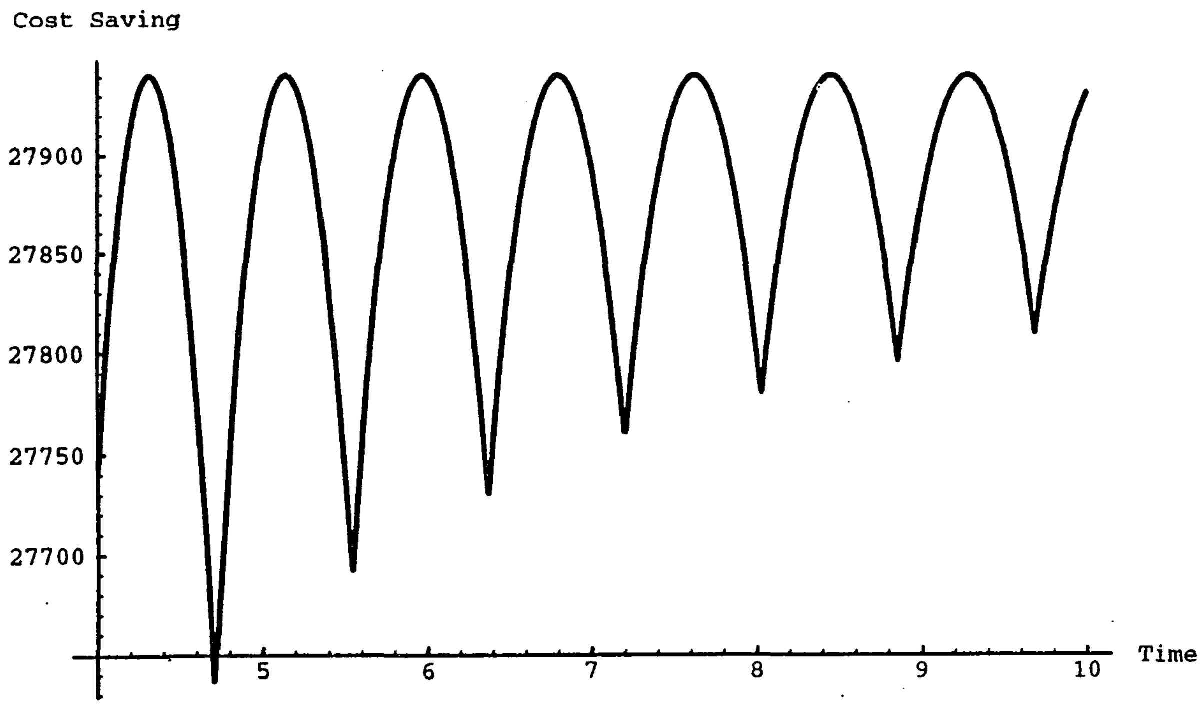


Figure 4. Cost saving vs. t_x

If $\lfloor\frac{R(t_b-t_o)}{Q_o}\rfloor \neq \lfloor\frac{R(t_e-t_o)}{Q_o}\rfloor$, then $t_x^*=t_o+\lfloor\frac{R(t_e-t_o)}{Q_o}\rfloor(\frac{Q_o}{R})$ and $n^* = \lfloor\frac{R(t_e-t_o)}{Q_o}\rfloor$.

Else

If $n(t_e)=\underline{n}$ (which implies $n(t_b)=\underline{n}$), then $t_x^*=t_b$ and $n^*=\underline{n}$.

Else if $n(t_b)=\bar{n}$ (which implies $n(t_e)=\bar{n}$), then $t_x^*=t_e$ and $n^*=\bar{n}$.

Else

if $G_4(t_b, \underline{n}) > G_4(t_e, \bar{n})$ then $t_x^* = t_b$ and $n^*= \underline{n}$,

else $t_x^* = t_e$ and $n^* = \bar{n}$.

Exhibit 2. The decision-making rule for "One Special Order" Policy

5.2 Two Special Orders Policy

In this subsection, we will examine the strategy of placing n equal-size orders from t_o to t_x, a special order $R(t_e - t_x)$ to meet the exact demand from t_x to t_e and an additional special order Q_s at t_e. As in the preceding sections, we will employ the "Non-Response" policy as the benchmark. By considering the duration from t_o to $(t_e + \frac{Q_s}{R})$, we can easily obtain the cost saving of the "Two Special Orders" policy over the "Non-Response" policy as follows.

$$\begin{aligned}\underset{n,t_x,Q_s}{\text{Maximize}}\ G_6 &= [\frac{Q_s}{R} + (t_e-t_o)](PR + \sqrt{2CRPF}) - nC - \frac{RPF(t_x-t_o)^2}{2n} \\ &- PR(t_x-t_o) - C - \frac{R(P-d)F(t_e-t_x)^2}{2} - R(P-d)(t_e-t_x) \\ &- C - (P-d)Q_s - \frac{Q_s^2(P-d)F}{2} \qquad (20)\end{aligned}$$

subject to: $t_b \leq t_x < t_e$

n is an integer

As in "One-Special Order" policy, we note that the above objective function is for the case where there is no regular EOQ replenishment point during the sale. If there is a regular EOQ replenishment point during the sale, the the objective function becomes as follows.

$$\begin{aligned}\underset{n,t_x,Q_s}{\text{Maximize}}\ G_7 = &\left(\frac{Q_o}{R}\right)[(P-d)R + \sqrt{2CR(P-d)F}] + \left[\frac{Q_s-Q_o}{R} + (t_e-t_o)\right]\\ &(PR + \sqrt{2CRPF}) - nC - \frac{RPF(t_x-t_o)^2}{2n} - PR(t_x-t_o)\\ &- C - \frac{R(P-d)F(t_e-t_x)^2}{2} - R(P-d)(t_e-t_x) - C - (P-d)Q_s\\ &- \frac{Q_s^2(P-d)F}{2}\end{aligned} \tag{21}$$

We note that the difference between objective functions G_6 and G_7 is constant. Therefore, the first derivatives of G_6 and G_7 are identical. From the maximization of the above problems, the following first derivative with respect to Q_s can be easily obtained.

$$\frac{\partial G_6}{\partial Q_s} = \frac{\partial G_7}{\partial Q_s} = \frac{PR + \sqrt{2CRPF}}{R} - (P - d) - \frac{(P - d)FQ_s}{R} \tag{22}$$

By setting equation (22) to zero, the optimal Q_s can be obtained as follows.

$$\begin{aligned}Q^*_{sG6}\ (= Q^*_{sG7} = Q^*_s) &= \frac{dR + \sqrt{2CRPF}}{(P - d)F} = \frac{PR + \sqrt{2CRPF}}{(P - d)F} - \frac{R}{F}\\ &= \frac{Rd}{(P-d)F} + \frac{P}{P-d}Q_o\end{aligned} \tag{23}$$

We note that the expression of Q^*_s in (23) is identical to the special order quantity shown in (5). We also note that the special order quantity is independent from the decision variables n and t_x.

For the computational convenience, we will ignore the two constraints that $t_b \leq t_x < t_e$ and n is an integer at the beginning and then reconsider them as we proceed. In such a case, we can have the following derivatives with respect to t_x and n.

$$\frac{\partial G_6}{\partial t_x} = \frac{\partial G_7}{\partial t_x} = - \frac{RPF(t_x - t_o)}{n} - PR + (t_e - t_x)R(P-d)F + R(P-d) \tag{24}$$

$$\frac{\partial G_6}{\partial n} = \frac{\partial G_7}{\partial n} = - C + \frac{RPF(t_x - t_o)^2}{2n^2} \tag{25}$$

By setting equation (25) equal to zero, the following relation can be easily obtained.

$$Q_o = (\frac{2CR}{PF})^{0.5} = \frac{(t_x - t_o)R}{n} \tag{26}$$

By substituting (26) into (24), it can be easily found that $\frac{\partial G_6}{\partial t_x} (= \frac{\partial G_7}{\partial t_x})$ < 0. This implies that the possibility for $\frac{\partial G_6}{\partial t_x}$ $(= \frac{\partial G_7}{\partial t_x}) = 0$ exists

---

If $n(t_e) = n(t_b) = \underline{n}$, then $t_x^* = t_b$, a special order to meet the demand from t_b to t_e, and an additional special order Q_s^* at t_e.

Else let $n = \bar{n}$, calculate t_x

  if $t_b \le t_x < t_e$ and $n(t_x) = \bar{n}$, then a special order to meet the demand from t_x^* to t_e, and an additional special order Q_s^* at t_e.

  else

    if $t_x < t_b$, then $t_x^* = t_b$, a special order to meet the demand from t_b to t_e, and an additional special order Q_s at t_e.

    else the strategy of placing two special orders during the sale is never optimal.

Exhibit 3. The decision-making rule for "Two Special Orders" policy

when $n = \bar{n}$ (because $n = \underline{n}$ will result in $\frac{\partial G_6}{\partial t_x}$ $(= \frac{\partial G_7}{\partial t_x}) < 0)$. Given $n = \bar{n}$, by rearranging equation (24), we can easily obtain t_x as follows.

$$t_x = \frac{Ft_e(P - d) + (PFt_o/\bar{n}) - d}{F(P + (P/\bar{n}) - d)}$$

We also note that t_x should meet the constraint $t_b \leq t_x < t_e$ and $n(t_x) = \bar{n}$. In summary, we can conclude the decision-making rule shown in Exhibit 3 for the case of two special orders during the sale.

5.3 Decision Process for the set $t_a \leq t_o < t_b < t_e$

In this section, we fist consider the case that there is no regular EOQ replenishment point exists during the sale (i.e., $\lfloor \frac{R(t_b - t_o)}{Q_o} \rfloor = \lfloor \frac{R(t_e - t_o)}{Q_o} \rfloor$). According to Exhibit 2 and Exhibit 3 in the sections 5.1 and 5.2, the potential optimal policies can be classified into the following five mutually exclusive and exhaustive cases. Among these five mutually exclusive and exhaustive cases, the first two cases are "One Special Order" policies, the third and the fourth cases are "Two Special Orders" policies, and the fifth case is "Non-Response" policy.

Case 1): $n(t_b)$ equal-size orders of $\frac{R(t_b - t_o)}{n(t_b)}$ to meet the demand from t_o to t_b, then a special order Q_s at t_b.

Case 2): $n(t_e)$ equal-size orders of $\frac{R(t_e - t_o)}{n(t_e)}$ to meet the demand from t_o to t_e, then a special order Q_s at t_e.

Case 3): $n(t_b)$ equal-size orders of $\frac{R(t_b - t_o)}{n(t_b)}$ to meet the demand from t_o to t_b, a special order of $R(t_e - t_b)$ to meet the demand from t_b to t_e, then an additional special order Q_s at t_e.

Case 4): $n(t_x)$ equal-size orders of $\frac{R(t_x - t_o)}{n(t_b)}$ to meet the demand from t_o to t_x, a special order of $R(t_e - t_x)$ to meet the demand from t_x to t_e, then an additional special order Q_s at t_e.

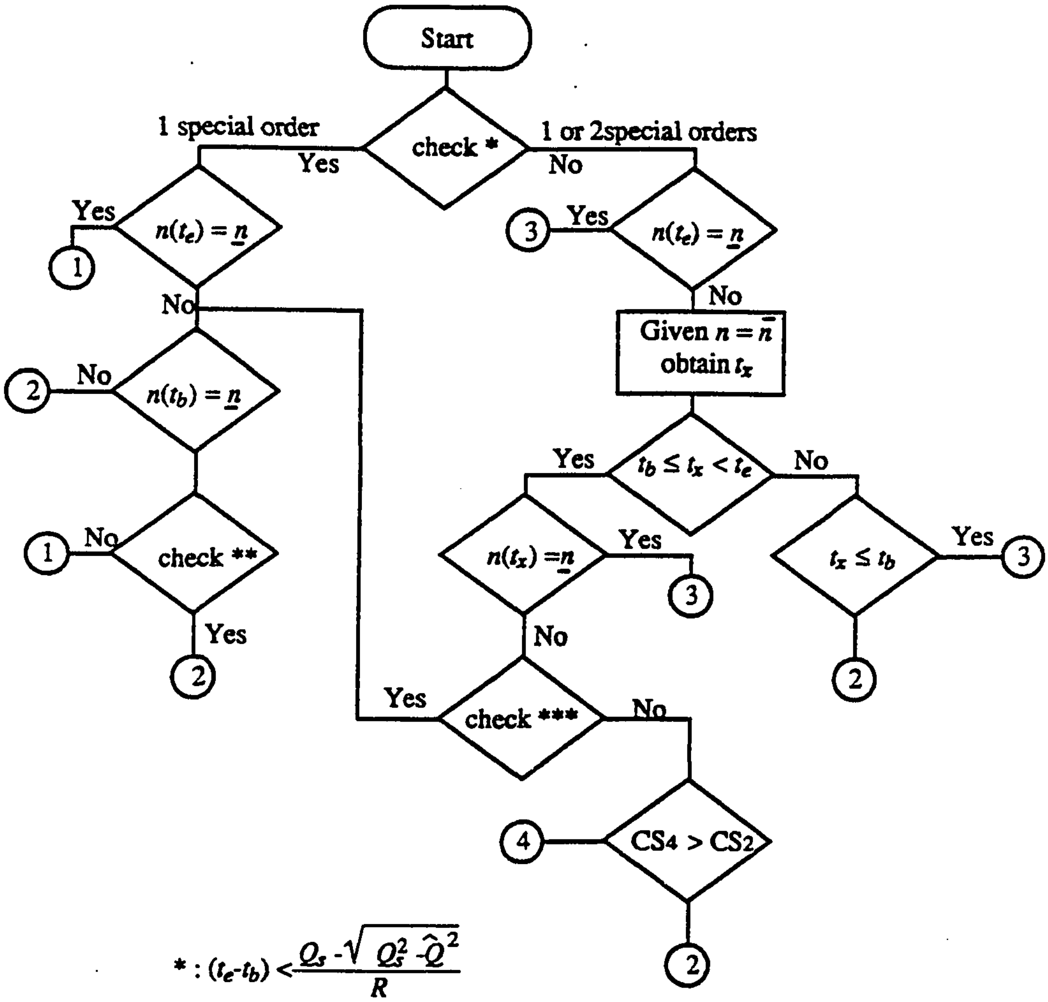
Case 5): "Non-Response" policy.

By employing the Exhibit 2 and Exhibit 3, we can have the decision process tree as shown in Diagram 1.

We now proceed to consider the case that there is one regular EOQ replenishment point exists during the sale (i.e., $\lfloor\frac{R(t_b - t_o)}{Q_o}\rfloor \neq \lfloor\frac{R(t_e - t_o)}{Q_o}\rfloor$). According to Exhibit 2 and Exhibit 3 in the sections 5.1 and 5.2, the potential optimal policies can be classified into the following three mutually exclusive and exhaustive cases. Among these three mutually exclusive and exhaustive cases, the first case are "One Special Order" policy, the second and the third are "Two Special Orders" policies. In this case, "Non-Response" policy will never be an optimal policy.

Case A): $\lfloor\frac{R(t_e - t_o)}{Q_o}\rfloor$ equal-size order of Q_o to meet the demand from t_o to t_x, where $t_x = t_o + \lfloor\frac{R(t_e - t_o)}{Q_o}\rfloor(\frac{Q_o}{R})$, then a special order Q_s at t_x

Case B): $n(t_b)$ equal-size orders of $\frac{R(t_b - t_o)}{n(t_b)}$ to meet the demand from t_o to t_b, a special order of $R(t_e - t_b)$ to meet the demand from t_b to t_e, then an additional special order Q_s at t_e.



$*$: $(t_e\text{-}t_b) < \dfrac{Q_s - \sqrt{Q_s^2 - \hat{Q}^2}}{R}$

$**$: $(t_e\text{-}t_b)\sqrt{2CRPF} - C \quad \dfrac{(t_e\text{-}t_o)^2RPF}{2(n+1)} + \dfrac{(t_b\text{-}t_o)^2RPF}{2n}$

$***$: $(t_e\text{-}t_x) < \dfrac{Q_s - \sqrt{Q_s^2 - \hat{Q}^2}}{R}$

i : check CSi > 0 or not, if Yes case *i* is optimal, if not "Non-Response" is optimal

Diagram 1. Decision process - no regular EOQ replenishment point during the sale

Case C): $n(t_x)$ equal-size orders of $\frac{R(t_x - t_o)}{n(t_b)}$ to meet the demand from t_o to t_x, a special order of $R(t_e - t_x)$ to meet the demand from t_x to t_e, then an additional special order Q_s at t_e.

By employing the Exhibit 2 and Exhibit 3, we can have the decision process tree as shown in Diagram 2.

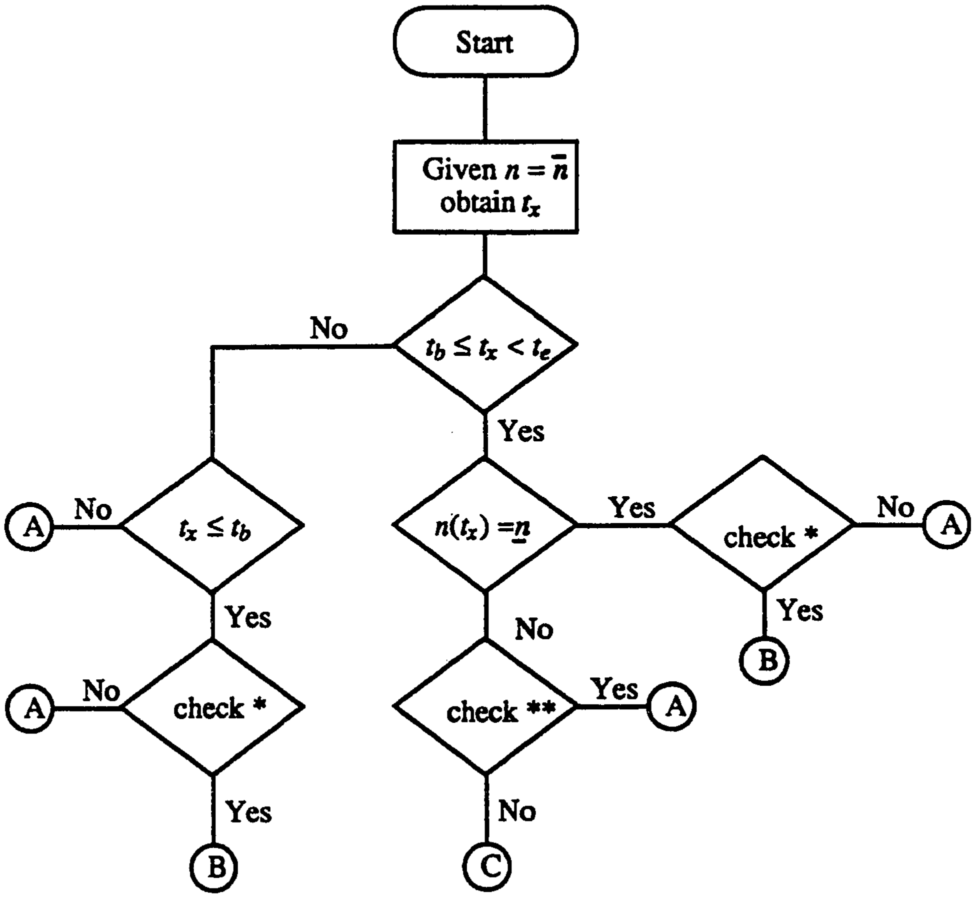
6. Numerical Results

In this section, we demonstrate that all eight cases for the set that $t_a \leq t_o < t_b < t_e$ (five for the case that no regular EOQ replenishment point during the sale and three for the case that one regular EOQ replenishment point during the sale) can be optimal policies with only t_o, t_b or t_e changes. Example 1 is designed to study the case that there is no regular EOQ replenishment point during the sale while Example 2 is to study the case that there is one regular EOQ replenishment point. The following values of the parameters are employed for both Example 1 and Example 2: $P = 100$, $R = 800$, $C = 8000$, $F = 0.2$. We note that the economic order quantity $Q_o = 800$ and the replenishment cycle is 1.

Example 1. Let t_b and t_e vary within the range of $5 < t_b < t_e < 6$ and t_o vary from 1 to 5. The results is shown in Table 1.

By examining Table 1 carefully, we make the following interesting observations for Example 1.

1) When t_b and t_e are sufficiently large (e.g., $(t_b, t_e) = (5.60, 5.75)$, $(5.60, 5.90)$ or $(5.75, 5.90)$), the policy of Case 3 is optimal regardless the values of t_o.



$$* : (t_e - t_b) < \frac{Q_s - \sqrt{Q_s^2 - \hat{Q}^2}}{R}$$

$$** : (t_e - t_x) < \frac{Q_s - \sqrt{Q_s^2 - \hat{Q}^2}}{R}$$

(i) : case i is optimal

Diagram 2. Decision process - one regular EOQ replenishment point during the sale

Table 1. No regular EOQ replenishment point during the sale

t_b	t_e	$t_o=1$		$t_o=2$		$t_o=3$		$t_o=4$		$t_o=5$	
5.15	5.30	1	11955	1	11940	1	11910	1	11820	3	8080
5.15	5.45	1	11955	1	11940	1	11910	1	11820	3	9580
5.15	5.60	2	13297	2	13282	2	13252	2	13162	3	10720
5.15	5.75	2	15763	2	15748	2	15718	2	15628	3	11500
5.15	5.90	2	17905	2	17890	2	17860	2	17770	4	12173
5.30	5.45	1	11820	1	11760	1	11640	1	11280	3	9580
5.30	5.60	1	11820	1	11760	1	11640	3	11360	3	10720
5.30	5.75	2	13162	2	13102	2	12982	2	12622	3	11500
5.30	5.90	2	15628	2	15568	2	15448	2	15088	3	11920
5.45	5.60	3	11744	3	11680	3	11573	3	11360	3	10720
5.45	5.75	3	11900	3	11875	3	11833	3	11750	3	11500
5.45	5.90	2	12937	2	12802	2	12535	2	12132	3	11920
5.60	5.75	3	11900	3	11875	3	11833	3	11750	3	11500
5.60	5.90	3	11984	3	11980	3	11973	3	11960	3	11920
5.75	5.90	3	11984	3	11980	3	11973	3	11960	3	11920

Table 2. One regular EOQ replenishment point during the sale

t_b	t_e	$t_o=1$		$t_o=2$		$t_o=3$		$t_o=4$		$t_o=5$	
5.25	6.15	B	10843	B	10801	B	10718	B	10468	C	6483
5.40	6.15	B	8830	B	8723	B	8510	B	7870	B	6270
5.40	6.30	B	10648	B	10541	B	10328	B	9688	B	8088
5.55	6.15	B	6683	B	6603	B	6468	B	6198	B	5388
5.55	6.30	B	8826	B	8745	B	8610	B	8340	B	7530
5.55	6.45	B	10644	B	10563	B	10428	B	10158	B	9348
5.70	6.15	B	4397	B	4362	B	4302	B	4182	B	3822
5.70	6.30	B	6864	B	6828	B	6768	B	6648	B	6288
5.70	6.45	B	9006	B	8970	B	8910	B	8790	B	8430
5.70	6.60	B	10824	B	10788	B	10728	B	10608	B	10248
5.85	6.15	A	3200	A	3200	A	3200	A	3200	A	3200
5.85	6.30	B	4506	B	4497	B	4482	B	4452	B	4361
5.85	6.45	B	6972	B	6963	B	6948	B	6918	B	6827
5.85	6.60	B	9114	B	9105	B	9090	B	9060	B	8970
5.85	6.75	B	10932	B	10923	B	10908	B	10878	B	10788

2) When t_0 = 1, 2, 3, or 4, the optimal policy can be Case 1, Case 2, or Case 3. On the other hand, When t_0 = 5, the optimal policy is either Case 3 or Case 4.

Also, it can be easily obtained that when d is sufficiently low (e.g., d = 1), the "Non-Response" policy is optimal.

Example 2. All the parameter values are the same as Example 1 with the exception that the values of t_b and t_e. In this case, there is a regular EOQ replenishment point occurs at 6 (i.e., $t_b < 6 < t_e$). The corresponding results are shown in Table 2.

By examining Table 2 carefully, we make the following interesting observations for Example 2.

1) Case A is optimal only if both t_b and t_e are sufficiently close to the regular EOQ replenishment point during the sale (e.g., (t_b, t_e) = (5.85, 6.15)).
2) When t_0 = 1, 2, 3, or 4, the optimal policy is either Case A or Case B. On the other hand, when t_0 = 5, the optimal policy can be Case A, Case B, or Case C.

Furthermore, we note that the following properties can be easily verified by way of simple calculus.

__Property 1.__ $\frac{\partial G}{\partial t_a} \leq 0$ and $\frac{\partial^2 G}{\partial t_a^2} \leq 0$ for all cases.

__Property 2.__ $\frac{\partial G}{\partial t_e} \geq 0$ for all cases.

Property 1 implies that the cost saving will increase when t_a is decreased. That is, if the buyer is informed the sale earlier, then the cost saving will be larger. In addition, the difference in the increase of the optimal cost saving decreases as t_a decreases.

Meanwhile, Property 2 implies that the duration of the sale increases, the optimal cost saving for the inventory policies will increase or remain the same.

7. Conclusions

In this paper, we constructed and analyzed an EOQ-type model for a buyer who was just informed of an announced temporary price decrease. For such a buyer, optimal inventory policies were derived by comparing cost savings of various cases. By analyzing the optimal inventory policies, several managerial insights were obtained. In particular, as the announcement time of the sale t_a is getting earlier (i.e., t_a is getting smaller), the optimal cost saving will increase or remain the same. On the other hand, as the duration of the sale increases, the optimal cost saving will increase or remain the same.

Several extensions can be made to enhance the basic models of this paper. For examples, as discussed in section 1 Introduction, it is assumed that the sale period is less than one regular EOQ cycle. By relaxing this assumption and allowing the sale period is greater than one regular EOQ model, interesting models that augment the models in this paper can be developed. Another class of extensions can be made with respect to the option of disposal. Implicitly, in this paper, it is assumed that the buyer does not have the option to dispose his on-hand inventory when the temporary sale is announced. It would be interesting to investigate the integration of inventory and disposal policies for announced temporary price decrease. We believe that such extensions will improve the applicability of inventory models in practice.

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Appendix. Proof of Lemma 1

__Lemma 1.__ The optimal time point at which the special order is placed for the set A: $t_a < t_b < t_e \leq t_o$ is at t_e.

Proof:

We will denote the on-hand inventory level at t_b by $\tilde{q}$ (i.e., $\tilde{q} = q - R(t_b - t_a)$). In addition, we define y to be the time interval between t_b and the time point at which the special order occurs. Also, we will denote the inventory level at $(t_b + y)$ after the special order (including the inventory before the special order) by Q_z. Figure 5 illustrates the "Response" policy and "Non-Response"policy. In order to measure the cost saving of the "Response" policy over the "Non-Response" policy accurately, the total cost will be calculated from t_b to $(t_b + y + \frac{Q_s}{R})$ (see e.g., Tersine, 1994). The total cost from the time point t_b to the time point $(t_e + y + \frac{Q_s}{R})$ for the "Response" policy, C_R, can be expressed as follows.

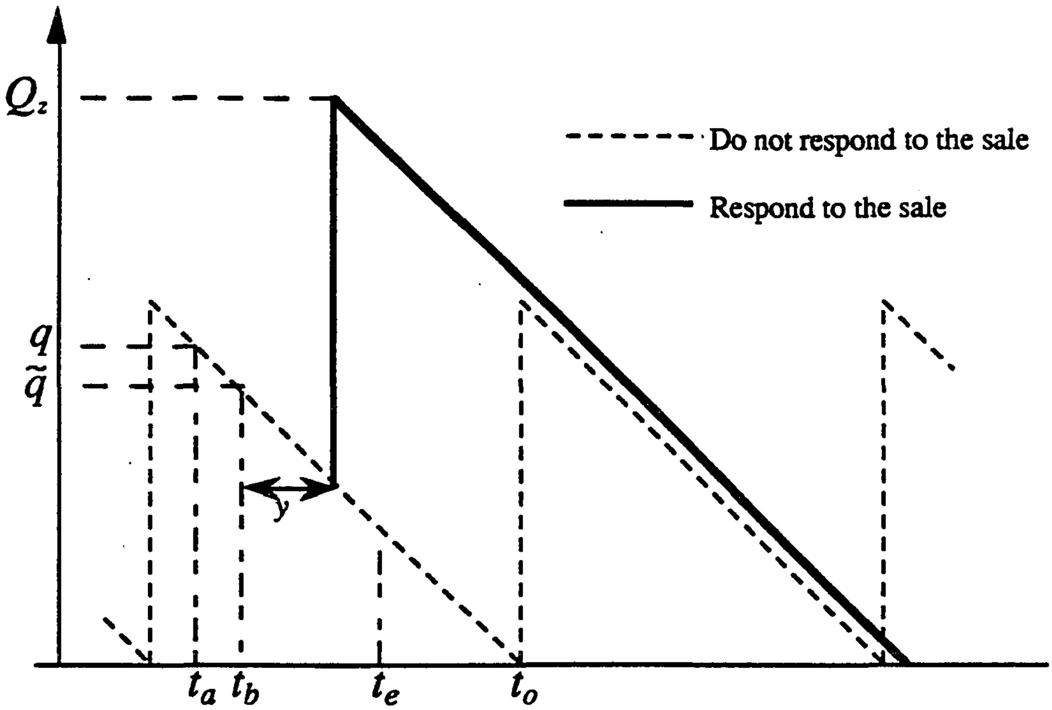

Figure 5. The general behavior for the case $t_a < t_b < t_e \leq t_o$

$$C_R = \frac{\tilde{q}^2 PF}{2R} + (Q_z - \tilde{q} + Ry)(P - d) + (Q_z - \tilde{q} + Ry)(\frac{\tilde{q} - Ry}{R})$$
$$(P - d)F + \frac{(Q_z - \tilde{q} + Ry)^2 (P - d)F}{2R} + C \tag{A.1}$$

The corresponding total cost for the same duration for the "Non-Response" policy, C_{NR}, is given by

$$C_{NR} = \frac{\tilde{q}^2 PF}{2R} + \frac{Q_z - \tilde{q} + Ry}{R} (PR + \sqrt{2CRPF}) \tag{A.2}$$

From the equations (A.1) and (A.2), the cost saving of the "Response" policy over the "Non-Response" policy is given by $G_0 = C_{NR} - C_R$. The objective now is to find the optimal y which will maximize G_0. Namely,

$$\underset{y}{\text{Maximize}}\ G_0 = C_{NR} - C_R \tag{A.3}$$

From the maximization of the above problem, the following first derivatives can be easily obtained.

$$\frac{\partial G_0}{\partial y} = (PR+\sqrt{2CRPF}) - R(P-d) + Q_z(P-d)F - 2(\tilde{q}-Ry)(P-d)F$$
$$- (Q_z - \tilde{q}+Ry)(P-d)F \tag{A.4}$$

By rearranging (A.4), we can obtain the following expression.

$$\frac{\partial G_0}{\partial y} = dR + ((\frac{2CR}{PF})^{0.5} - (\tilde{q}-Ry))PF + (\tilde{q}-Ry)dF > 0 \tag{A.5}$$

From (A.5), we note that $\partial G_0/\partial y > 0$. This implies that the cost saving will increase when y is increased. That is, if the buyer places a special order during the sale, his optimal strategy is to place the special order as late as possible. In such a case the optimal time point to place a special order is at time point t_e.

CHAPTER III.

OPTIMIZATION CRITERIA FOR INVENTORY-INVESTMENT IN SETUP OPERATIONS POLICIES: PROFIT VS. RETURN ON INVESTMENT

A paper to be submitted to *Decision Sciences*

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ABSTRACT

We construct and analyze optimal policies for inventory and investment in setup operations under profit maximization and under return on investment maximization. Under a general functional form of investment in setup operations, we derive the optimality conditions under profit maximization and under return on investment maximization. By comparing and contrasting the optimality conditions, several interesting economic implications are obtained. Also, for two specific functional forms of investment in setup operations (linear and hyperbolic), the closed-from optimal solutions and the decision making rules are derived. From the solution and rules, additional economic implications are obtained.

INTRODUCTION

In this paper, we construct and analyze inventory and investment in setup operations policies under profit maximization and return on investment maximization for decision makers of inventory systems. We assume that the option of investing in setup operations is available. We also assume that the return on investment is the ratio of the profit to the sum of the average inventory investment and the capital investment in setup operations. Under these assumptions, we formulate the inventory and investment in setup operations policies under both profit maximization and return on investment maximization and derive the optimality conditions. Also, several interesting economic implications at the optimality conditions are obtained. The primary contributions of our paper are: 1) A unique analytical formulation to examine the return on investment of the option of investing in setup operations, 2) Several interesting economic interpretations for the optimality conditions under profit maximization and return on investment maximization, and 3) Closed-from optimal solutions and the decision making rules when the setup cost function is linear or hyperbolic.

The idea of employing profits as a performance measure of inventory models has been explored as early as the 1950's (see e.g., Whitin [16] or Smith [14]). Ladany and Sternlieb [6] not only uses the profit levels as the performance measure, but also provides insights on relations among price, cost, and demand by making the demand dependent on the price and the price dependent on the cost and a fixed mark-up. Schroeder and Krishnan [13] proposes an inventory model under an alternative optimization criterion of return on investment inventory maximization. Morse and Scheiner [8] investigates inventory models under three alternative criteria which are cost minimization, return on investment and residual income. Subsequently, Arcelus and Srinivasan [1] [2] compare and contrast profit maximization vs. return on inventory investment maximization with respect to constant elasticity demand functions. Also, Rosenberg [12] compares and contrasts profit maximization vs. return on inventory investment with respect to logarithmic concave demand functions. In his analysis, under linear demand functions, closed-form optimal solutions are employed for the return on inventory investment model while an examination of optimality conditions and an iterative procedure (e.g, the Newton-Raphson method) are employed for the profit maximization model. In contrast to the iterative procedure under linear demand function for profit maximization model proposed by Rosenberg [12], Chen and Min [4] derives the optimal closed-form solution for both profit maximization and return on inventory investment maximization under linear demand functions. Also, a comprehensively comparative analysis is presented in Chen and Min [4] for both profit maximization and return on inventory investment maximization models.

Recently, the superiority of an inventory management system called Zero Inventory (often synonymous with Kanban and Just-in-Time; see e.g., Zangwill [17]) has attracted a great deal of attention not only from industries but also from the academia. The essential philosophy of Zero Inventory management system is that the inventory results from operational inefficiencies. Hence, the higher the level of inventory, the greater the operational inefficiencies. From this perspective, it is well known that several Japanese and American producers strive to reduce the level of inventory as much as possible. In order to reduce the level of inventory, numerous experts in industry and academia find it essential to reduce the setup cost of production. In Porteus [9], such efforts to reduce the setup cost are mathematically incorporated by introducing an investment cost function of reducing the setup cost to undiscounted EOQ models. For the cases of logarithmic investment cost functions and power investment cost functions, his models demonstrate decreased operational costs when the setup cost is reduced. Porteus [10] extends Porteus [9] to the cases of discounted EOQ models. Billington [3] formulates a model of which setup cost is a function of capital expenses and investigates the relations among holding, setup, and capital expenses. Hong, Xu, and Hayya [5] proposes a dynamic lot-sizing model of which setup reduction and process quality are functions of capital expenditure. We note that, in all these papers, the performance criterion has been the minimization of operational costs (as compared to the maximizations of profit and return on investment in our models). We also note that there have been few analytical model that examines the return of the investment in improving the setup operations.

In this paper, we construct and analyze inventory and investment in setup operations policies by employing the profit maximization and the return on investment maximization as the performance criteria. By treating inventory and capital expenditure in reducing setup operations as investments for the purpose of generating profits, the return on investment is defined to be the ratio of profit per unit time to the sum of the average inventory investment per unit time and the investment in reducing setup operations per unit time. The decision variables of our models are the economic order quantity and the desired level of the investment in reducing setup operations. We formulate the models and derive and interpret the optimality conditions for general setup cost function. For specific cases of linear and hyperbolic setup cost functions, it is shown that the optimal investment decisions for the linear setup cost case is more sensitive than the optimal investment decisions for the hyperbolic setup cost case.

The rest of this paper is organized as follows. We first formulate the inventory and investment in setup operations models and derive and interpret their optimality conditions. Next, for the specific cases of the linear setup cost and the hyperbolic setup cost, the optimal closed-form solutions are obtained and several interesting managerial insights are presented. Finally, summary and concluding remarks are made.

DEFINITIONS AND ASSUMPTIONS

Throughout this paper, for the decision maker of the inventory system, the following notations and definitions will be employed.

Q: the order quantity per order. Unit: units.

c: the variable cost per unit. i.e., the unit cost. Unit: \$/unit.

i: the inventory holding cost expressed as a fraction of the unit cost per unit time, excluding the opportunity cost of funds tied up in inventory. Unit: 1/unit time.

I: the inventory holding cost expressed as a fraction of the unit cost per unit time, including the opportunity cost of funds tied up in inventory. Unit: 1/unit time.

i_{oc}: the opportunity cost (or the cost of capital), $I = i + i_{oc}$. Unit: 1/unit time.

K: the amount of capital investment in setup operations (to be invested per unit time). $K_{min} \leq K \leq K_{max}$. Unit: \$/unit time.

S(K): the setup cost per order as a function of K. Unit: \$.

p: the selling price per unit. Unit: \$/unit.

d: the sales quantity per unit time. Unit: units/unit time.

The basic assumptions of our models are: 1) There are no learning effects in setup or production; 2) Shortages or delivery lags is not allowed; 3) The sales quantity per unit time as well as the selling price per unit are deterministic and constant over time; 4) the setup cost function S(K) is strictly decreasing with respect to K.

PROFIT MAXIMIZATION MODEL

Given the above definitions and assumptions, we first develop a profit maximization model with an option to invest in setup operations as follows. The revenue per unit time is given by pd. And the corresponding per unit time setup cost, total variable cost, inventory holding cost, and the amount of investment in setup operations are given by $\frac{S(K)d}{Q}$, cd, $\frac{IcQ}{2}$, and K, respectively. The total cost per unit time, TC, is:

$$TC = \frac{S(K)d}{Q} + cd + \frac{IcQ}{2} + K \tag{1}$$

And the corresponding profit per unit time, π, is:

$$\pi = pd - \frac{S(K)d}{Q} - cd - \frac{IcQ}{2} - K \qquad (2)$$

The objective of the decision maker is to maximize π over Q and K subject to $K_{min} \leq K \leq K_{max}$. An equivalent standard form (see e.g., Luenberger [7]) for this problem is:

$$\text{Problem 1:} \quad \underset{Q,K}{\text{Minimize}} \; -\pi$$
$$\text{s.t.} \quad K_{min} - K \leq 0$$
$$K - K_{max} \leq 0$$

The corresponding Lagrangian function, L_π, is given by $L_\pi = -\pi + \lambda_1(K_{min} - K) + \lambda_2(K - K_{max})$. The first order necessary conditions for Problem 1 are:

$$\frac{\partial L_\pi}{\partial Q} = - S(K)d/Q^2 + Ic/2 = 0 \qquad (3)$$

$$\frac{\partial L_\pi}{\partial K} = S'(K)d/Q + 1 - \lambda_1 + \lambda_2 = 0 \qquad (4)$$

$$\lambda_1(K_{min} - K) = 0 \qquad (5)$$

$$\lambda_2(K - K_{max}) = 0 \qquad (6)$$

$$\lambda_1 \geq 0 \qquad (7)$$

$$\lambda_2 \geq 0 \qquad (8)$$

$$K_{min} - K \leq 0 \qquad (9)$$

$$K - K_{max} \leq 0 \qquad (10)$$

If an optimal solution (Q, K) is such that $K = K_{min}$, then the first order necessary conditions are reduced to:

$$Q = (2S(K_{min})d/(Ic))^{0.5}, \; \lambda_1 = S'(K_{min})d/Q + 1 \geq 0, \text{ and } \lambda_2 = 0 \qquad (11)$$

We note that the corresponding second order sufficient condition is satisfied for any pair of (Q, K) which satisfies the first order necessary conditions.

On the other hand, if an optimal solution (Q, K) is such that K =

K_{max}, then the first order necessary conditions are reduced to:

$$Q = (2S(K_{max})d/(Ic))^{0.5},\ \lambda_1 = 0, \text{ and } \lambda_2 = -S'(K_{max})d/Q - 1 \geq 0 \quad (12)$$

We note that the corresponding second order sufficient condition is satisfied for any pair of (Q, K) which satisfies the first order necessary conditions.

Thus far, we have examined the optimality conditions of the boundary optimal solutions (in the sense that $K = K_{min}$ or $K = K_{max}$). We now proceed to examine the optimality conditions of the interior optimal solutions (in the sense that $K_{min} < K < K_{max}$). If an optimal solution (Q, K) is such that $K \in (K_{min}, K_{max})$, then the first order necessary conditions are: $\lambda_1 = \lambda_2 = 0$, and

$$\frac{\partial \pi}{\partial Q} = S(K)d/Q^2 - Ic/2 = 0 \tag{13}$$

$$\frac{\partial \pi}{\partial K} = -S'(K)d/Q - 1 = 0 \tag{14}$$

From (13) and (14), the Hessian matrix of π, H_π, is given by

$$H_\pi = \begin{bmatrix} -2S(K)d/Q^3 & S'(K)d/Q^2 \\ S'(K)d/Q^2 & -S''(K)d/Q \end{bmatrix} \tag{15}$$

From (15), the corresponding second order sufficient condition becomes

$$2S(K)S''(K) > (S'(K))^2 \tag{16}$$

We note that the first order necessary conditions and the second order sufficient condition of the profit maximization problem are equivalent to those of the cost minimization problem. Finally, we note that both boundary and interior solutions are only local optimal solutions. Because the functional form of S(K) is not specified in our model (i.e., broad classes of functional forms are admissible), the issue of global optimality is difficult to address. The extensive analyses of local boundary and interior optimal solutions in later sections, however, are valuable because: 1) the global optimal solution is an element of the set of local optimal solutions, and 2) the global optimal solution may change from one local optimal solution to another local optimal solution even with a minor perturbation in the values of parameters.

RETURN ON INVESTMENT MAXIMIZATION MODEL

Thus far, we have developed a profit maximization model and characterized the corresponding optimal solutions. Let us now develop a return on investment maximization model. In the literature of inventory theory, there have been numerous papers on the return on inventory investment. e.g., Schroeder and Krishnan [13], Morse and Scheiner [8] Arcelus and Srinivasan [1] [2], or Rosenberg [12]. In these papers, the inventory is viewed as a capital investment for profits. From this perspective, the return on inventory investment is defined to be the ratio of the profit per unit time to the inventory investment per unit time. An additional distinction of the return on inventory investment is that the inventory holding cost rate I is now replaced by i, which is exclusive of opportunity costs. The reason for this change is that, because the decision maker wants to maximize the return on investment, it is no longer appropriate to pre-specify a return on capital in the inventory holding cost.

In the return on inventory investment models, the inventory is viewed as a capital investment for profits. Let us now suppose that K is invested in setup operations per unit time. Then, this capital expenditure must also be viewed as an investment for the same purpose. That is, both inventory investment and investment in setup operations must be viewed in the same way based on their profitability. Hence, in addition to the inventory investment per unit time of $cQ/2$, if a per unit time capital investment of K is made for the setup operations, the total capital investment per unit time is equal to $cQ/2 + K$. The corresponding profit is given by

$$P = pd - \frac{S(K)d}{Q} - cd - \frac{icQ}{2} - K \tag{17}$$

Therefore, the return on combined capital investment, R, is given by

$$R = (pd - \frac{S(K)d}{Q} - cd - \frac{icQ}{2} - K)/(\frac{cQ}{2} + K) \tag{18}$$

The objective of the decision maker is to maximize R over Q and K subject to $K_{min} \leq K \leq K_{Max}$. An equivalent standard form (see e.g., Luenberger [7]) for this problem is :

$$\text{Problem 2:} \quad \underset{Q,K}{\text{Minimize}} \quad -R$$
$$\text{s.t.} \quad K_{min} - K \leq 0$$
$$K - K_{Max} \leq 0$$

The corresponding Lagrangian function, L_R, is given by $L_R = -R + \mu_1(K_{min} - K) + \mu_2(K - K_{Max})$. The first order necessary conditions for Problem 2 are:

$$\frac{\partial L_R}{\partial Q} = -[(S(K)d/Q^2 - ic/2)(cQ/2 + K) - cP/2]/(cQ/2 + K)^2 = 0 \tag{19}$$

$$\frac{\partial L_R}{\partial K} = -[(-S'(K)d/Q - 1)(cQ/2 + K) - P]/(cQ/2 + K)^2 - \mu_1 + \mu_2 = 0 \tag{20}$$

$$\mu_1(K_{min} - K) = 0 \tag{21}$$

$$\mu_2(K - K_{max}) = 0 \tag{22}$$

$$\mu_1 \geq 0 \tag{23}$$

$$\mu_2 \geq 0 \tag{24}$$

$$K_{min} - K \leq 0 \tag{25}$$

$$K - K_{max} \leq 0 \tag{26}$$

If an optimal solution (Q, K) is such that $K = K_{min}$, then the first order necessary conditions are reduced to:

$$[S(K_{min})d/Q^2 - ic/2](cQ/2 + K_{min}) = cP/2, \ \mu_2 = 0, \text{ and}$$

$$\mu_1 = [(S'(K_{min})d/Q + 1)(cQ/2 + K_{min}) + P]/(cQ/2 + K_{min})^2 \geq 0 \tag{27}$$

We note that the corresponding second order sufficient condition is satisfied for any pair of (Q, K) which satisfies the first order necessary conditions.

On the other hand, if an optimal solution (Q, K) is such that $K = K_{max}$, then the first order necessary conditions are reduced to:

$$[S(K_{max})d/Q^2 - ic/2](cQ/2 + K_{max}) = cP/2, \ \mu_1 = 0, \text{ and}$$

$$\mu_2 = [(-S'(K_{max})d/Q - 1)(cQ/2 + K_{max}) - P]/(cQ/2 + K_{max})^2 \geq 0 \tag{28}$$

We note that the corresponding second order sufficient condition is satisfied for any pair of (Q, K) which satisfies the first order necessary conditions.

Thus far, we have examined the optimality conditions of the boundary optimal solutions (in the sense that $K = K_{min}$ or $K = K_{max}$). We now proceed to examine the optimality conditions of the interior optimal solutions (in the sense that $K_{min} < K < K_{max}$). If an optimal solution (Q, K) is such that $K \in (K_{min}, K_{max})$, then the first order necessary conditions are:

$$\mu_1 = \mu_2 = 0, \text{ and}$$

$$\frac{\partial R}{\partial Q} = [(S(K)d/Q^2 - ic/2)(cQ/2 + K) - cP/2]/(cQ/2 + K)^2 = 0 \tag{29}$$

$$\frac{\partial R}{\partial K} = [(-S'(K)d/Q - 1)(cQ/2 + K) - P]/(cQ/2 + K)^2 = 0 \tag{30}$$

We can obtain the Hessian matrix of R evaluated at a solution (Q, K) of equations (29) and (30), H_R, as below.

$$H_R = \frac{-1}{cQ/2 + K}\begin{bmatrix} 2S(K)d/Q^3 & -S'(K)d/Q^2 \\ -S'(K)d/Q^2 & S''(K)d/Q \end{bmatrix} \tag{31}$$

From (31), the corresponding second order sufficient condition becomes

$$2S(K)S''(K) > (S'(K))^2 \tag{32}$$

We note that the functional forms of the second order sufficient conditions of both Problem 1 and Problem 2 are identical (even though the optimal values of K may be different).

OPTIMALITY ANALYSIS

In this section, we analyze the optimal solutions and derive interesting managerial insights. First, we examine the interior optimal solution cases, followed by the boundary optimal solution cases. Next, we investigate the relative magnitudes of optimal solutions under profit maximization and under return on investment maximization. Let (Q_π, K_π) and (Q_R, K_R) denote the optimal solutions under profit maximization and under return on investment maximization, respectively.

We begin our analysis by examining the profit maximization model. By rearranging the optimality condition (13), we have

$$S(K_\pi)d/Q_\pi = IcQ_\pi/2 \tag{33}$$

The economic interpretation of (33) is that the setup cost per unit time is equal to the inventory holding cost (including the opportunity cost) per unit time at the optimality. Also by rearranging the optimality condition (14), we have

$$-S'(K_\pi)d/Q_\pi = 1 \tag{34}$$

We can view $-S'(K_\pi)$ as the marginal setup cost saving. Also, we can view $\frac{\partial K}{\partial K} = 1$ as the marginal investment per unit time in the setup operations. Hence, at the optimality, the marginal setup cost saving per unit time is equal to the marginal investment in setup operations per unit time.

Let us now examine the return on investment maximization problem. By rearranging the optimality condition (29), we have

$$S(K_R)d/Q_R = icQ_R/2 + (cQ_R/2)R \tag{35}$$

The economic interpretation of (35) is that, at the optimality, the setup cost per unit time is equal to the inventory holding cost (excluding the opportunity cost) per unit time plus the portion of the profit per unit time which is contributed by the inventory investment.

Also by rearranging the optimality condition (30), we have

$$-S'(K_R)d/Q_R = 1 + R \tag{36}$$

The economic interpretation of (36) is that, at the optimality, the marginal setup cost saving per unit time is equal to the marginal investment per unit time in the setup operations plus the return on investment per unit time. That is, the marginal setup cost saving per unit time is strictly larger than the marginal investment per unit time in the setup operations at the optimality.

In addition, by rearranging terms in the optimality conditions (29) and (30), we can obtain the following expression for Q_R:

$$Q_R = (2S(K) - S'(K)K)/(p - c) \tag{37}$$

We note that this is a generalized expression of $Q = 2S/(p-c)$ in Schroeder and Krishnan [13] where the option to invest in the setup operations is not available.

We now proceed to examine the boundary optimal solution cases. From (11), when $(Q_\pi, K_\pi) = ((2S(K_{min})d/(Ic))^{0.5}, K_{min})$, then $-S'(K_{min})d/Q_\pi \leq 1$. i.e., the marginal setup cost saving per unit time is less than or equal to the marginal investment per unit time in the setup operations. Also, from (12), when $(Q_\pi, K_\pi) = ((2S(K_{max})d/(Ic))^{0.5}, K_{max})$, then $-S'(K_{max})d/Q \geq 1$. i.e., the marginal setup cost saving per unit time is greater than or equal to the marginal investment per unit time in the setup operations.

In addition, from (27), when $(Q_R, K_R) = (\frac{c + (c^2+4E_{1,min}cK_{min})^{0.5}}{2E_1c}, K_{min})$ where $E_{1,min} = \frac{pd - cd - K_{min} + iK_{min}}{2S(K)d}$, then $-S'(K_{min})d/Q_R \leq 1 + R$. i.e., the marginal setup cost saving per unit time is less than or equal to the marginal investment per unit time in the setup operations plus the return on investment per unit time. Also, form (28), when $(Q_R, K_R) = (\frac{c + (c^2+4E_{1,max}cK_{max})^{0.5}}{2E_1c}, K_{max})$, where $E_{1,max} = \frac{pd - cd - K_{max} + iK_{max}}{2S(K)d}$, then $-S'(K_{max})d/Q_R \geq 1 + R$. i.e., the marginal setup cost saving per unit time is greater than or equal to the marginal investment per unit time in the setup operations plus the return on investment per unit time.

LINEAR SETUP COST CASE

In this section, we consider the case that the setup cost has a linear relation to the amount of investment for setup operations. i.e., $S(K) = \alpha - \beta K$ (see e.g., Billington [3]), where α and β are positive constants and α is the upper bound of the setup cost. For the profit maximization decision maker, the objective function and the constraints can be described as follows.

$$\underset{Q,\,K}{\text{Minimize}}\ -\pi = -pd + \frac{(\alpha-\beta K)d}{Q} + cd + \frac{IcQ}{2} + K$$

$$\text{s.t.}\ -K + K_{min} \leq 0$$

$$K - K_{max} \leq 0$$

For the return on investment maximization decision maker, the objective function and the constraints can be described as follows.

$$\underset{Q,\,K}{\text{Minimize}}\ -R = \frac{-pd + ((\alpha-\beta K)d/Q) + cd + (icQ/2) + K}{K + (cQ/2)}$$

$$\text{s.t.}\ -K + K_{min} \leq 0$$

$$K - K_{max} \leq 0$$

The corresponding boundary and interior solutions which are derived from the first order necessary conditions under both profit maximization and return on investment maximization are summarized in Table 1. Moreover, in Table 1, we also present whether the second order sufficient conditions are satisfied or not at the solutions, which are obtained from the first order necessary conditions.

From Table 1, we note that, for the linear setup cost case, the interior points are never optimal. The optimal solutions for both profit maximization and return on investment maximization always occurs at the boundary points. Hence, the optimal investment decision for setup operations under linear setup cost is either K_{min} or K_{max} regardless of the choice of optimization criterion. By comparing the objective function values evaluated at K_{min} and K_{max} under both profit maximization and return on investment maximization, the following decision-making rules can be developed.

	Boundary Solutions			Interior Solutions		
	K	Q	S.O.S.C.	K	Q	S.O.N.C.
Profit Maximization	K_{min} or K_{max}	$\sqrt{\frac{2(\alpha-\beta K)d}{Ic}}$	satisfied	$\frac{2\alpha-dIc\beta^2}{2\beta}$	$d\beta$	not satisfied
ROI Maximization	K_{min} or K_{max}	$\frac{c+\sqrt{c^2+4A_1cK}}{2A_1c}$	satisfied	$\frac{2\alpha-Q(p-c)}{\beta}$	$\frac{-A_2+\sqrt{A_2^2+2\alpha dA_3}}{A_3}$	not satisfied

Table 1. The solutions for linear setup cost case

Rule 1. If $Ic < \frac{K_{max} - K_{min}}{Q_{\pi,min} - Q_{\pi,max}}$ and $\frac{\alpha - \beta K_{min}}{Q^2_{R,min}} > \frac{\alpha - \beta K_{max}}{Q^2_{R,max}}$,

then $K^*_\pi = K^*_R = K_{min}$.

Rule 2. If $Ic > \frac{K_{max} - K_{min}}{Q_{\pi,min} - Q_{\pi,max}}$ and $\frac{\alpha - \beta K_{min}}{Q^2_{R,min}} < \frac{\alpha - \beta K_{max}}{Q^2_{R,max}}$,

then $K^*_\pi = K^*_R = K_{max}$.

Rule 3. If $Ic < \frac{K_{max} - K_{min}}{Q_{\pi,min} - Q_{\pi,max}}$ and $\frac{\alpha - \beta K_{min}}{Q^2_{R,min}} < \frac{\alpha - \beta K_{max}}{Q^2_{R,max}}$,

then $K^*_\pi = K_{min}$ and $K^*_R = K_{max}$.

Rule 4. If $Ic > \frac{K_{max} - K_{min}}{Q_{\pi,min} - Q_{\pi,max}}$ and $\frac{\alpha - \beta K_{min}}{Q^2_{R,min}} > \frac{\alpha - \beta K_{max}}{Q^2_{R,max}}$,

then $K^*_\pi = K_{max}$ and $K^*_R = K_{min}$.

We note that, in the above statements, $Q_{\pi,min} = (\frac{2d(\alpha-\beta K_{min})}{Ic})^{0.5}$, $Q_{\pi,max} = (\frac{2d(\alpha-\beta K_{max})}{Ic})^{0.5}$, $Q_{R,min} = \frac{c + (c^2+4A_{1,min}cK_{min})^{0.5}}{2A_1c}$, and $Q_{R,max} = \frac{c + (c^2+4A_{1,max}cK_{max})^{0.5}}{2A_1c}$, where $A_{1,min} = \frac{pd - cd - K_{min} + iK_{min}}{2(\alpha - \beta K_{min})d}$ and $A_{1,max} = \frac{pd - cd - K_{max} + iK_{max}}{2(\alpha - \beta K_{max})d}$.

From these decision-making rules, we note that the combination of the optimal investment decisions for setup operations under profit maximization and return on investment maximization are among the following four cases: 1) $K^*_\pi = K^*_R = K_{min}$, 2) $K^*_\pi = K^*_R = K_{max}$, 3) $K^*_\pi = K_{min}$; $K^*_R = K_{max}$, and 4) $K^*_\pi = K_{max}$; $K^*_R = K_{min}$. It can be easily observed that, under different optimization criteria for the linear setup cost case, the optimal investment decisions in the setup operations may be identical even though the optimal order quantities Q^*s are different. For example, Rule

1 and Rule 2 result in $K_\pi^* = K_R^* = K_{min}$ and $K_\pi^* = K_R^* = K_{max}$ while $Q_\pi^* \neq Q_R^*$ (i.e., Case 1 and Case 2). On the other hand, the optimal investment decisions may be in the opposite directions in the sense that $K_\pi^* = K_{min}$ and $K_R^* = K_{max}$ (or $K_\pi^* = K_{max}$ and $K_R^* = K_{min}$) (i.e., Case 3 and Case 4).

In order to illustrate the features of the linear setup cost case under both profit maximization and return on investment maximization, we provide the following numerical example.

Example 1. Suppose that $\alpha = 500$, $\beta = 1$, $d = 25$, $I = 0.2$, $c = 100$, $p = 150$, $K_{min} = 50$, and $K_{max} = 480$.

We plot the negative profit ($-\pi$) surface and the negative return on investment ($-R$) surface in Figure 1 and Figure 2, which show that the interior solutions derived from solving the first order necessary conditions (i.e., $(K_\pi, Q_\pi) = (25, 250)$ and $(K_R, Q_R) = (16.67, 166.67)$) are saddle points. By employing the decision-making rules developed earlier in this section and Table 1, the optimal solutions can be easily found at boundary points $(K_\pi^*, Q_\pi^*) = (480, 7.07)$ for profit maximization and $(K_R^*, Q_R^*) = (480, 3.11)$ for return on investment maximization. In this example, the investment decisions are identical for both profit maximization and return on investment maximization (i.e., case 2. $K_\pi^* = K_R^* = K_{max}$) even though the optimal order quantities Q^*s are different (i.e., $Q_\pi^* = 7.07$ and $Q_R^* = 3.11$).

We now demonstrate that all of those four cases can exist with only one or two changes in values of parameters. To achieve our objective, we select the per unit cost c and the sales quantity per unit time d as the parameters whose values change. Specifically, we let the per unit cost c

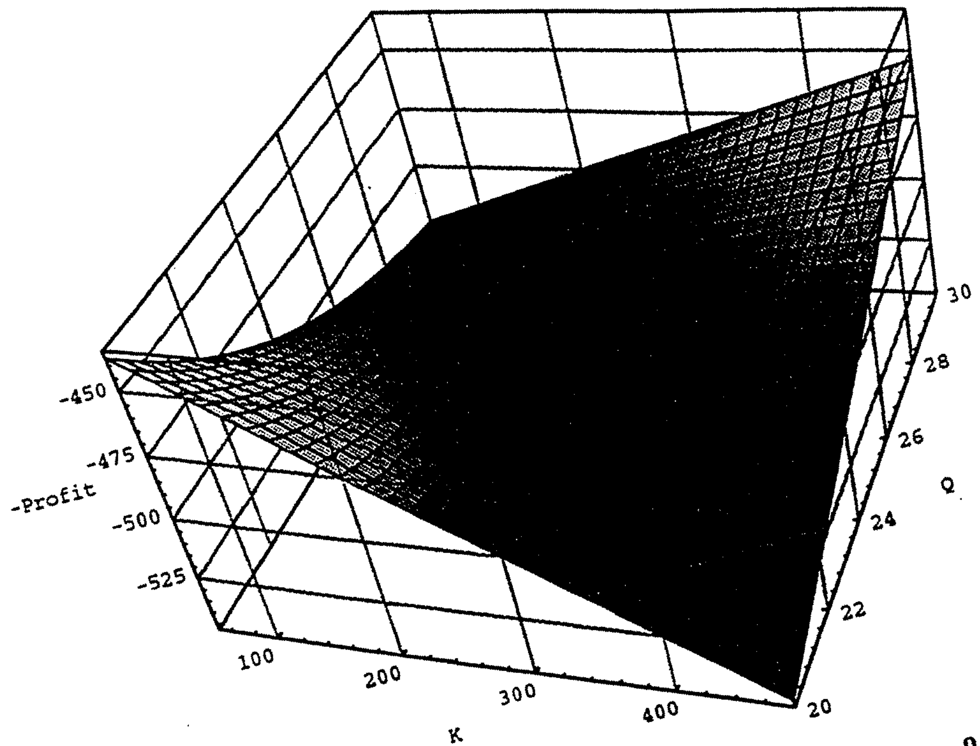

Figure 1. Negative profit $(-\pi)$ surface with respect to Q and K under linear setup cost function

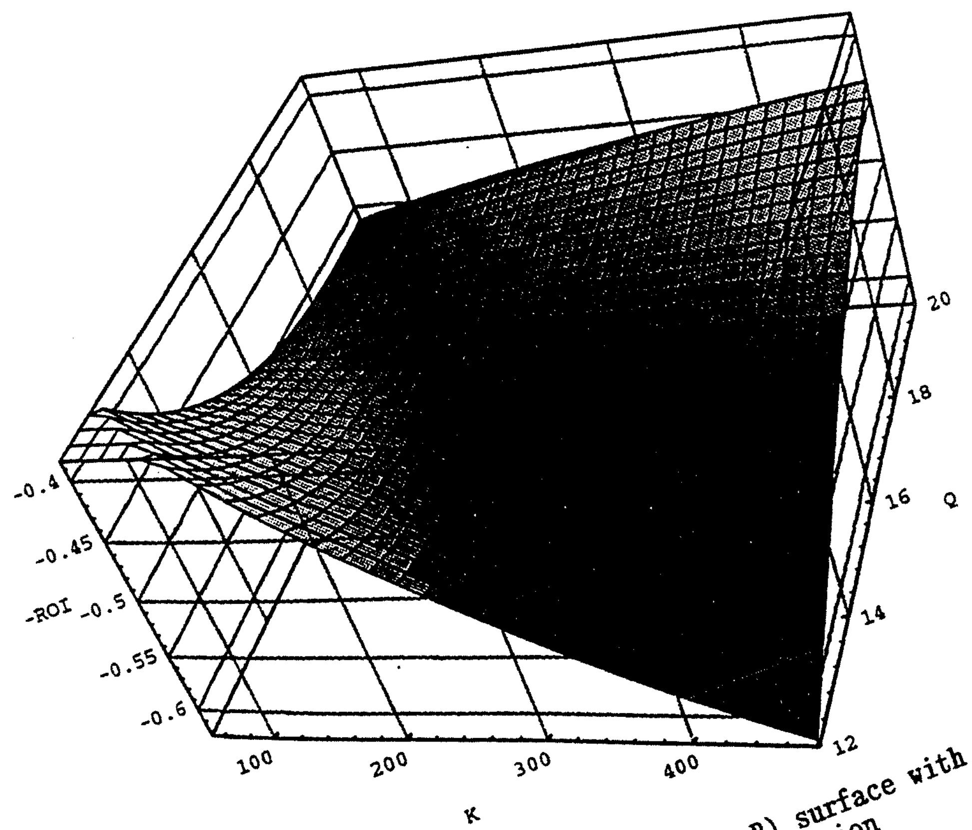

Figure 2. Negative return on investment (-R) surface with respect to Q and K under linear setup cost function

increases from 30 to 100 by step size 10 and the sales quantity per unit time d decreases from 50 to 15 by step size 5. The results are presented in Table 2.

By examining Table 2 carefully, we make the following interesting observations.

1) When (d, c) = (50, 30), the optimal investment decisions under profit maximization and return on investment maximization are identical at $K_{\pi}^{*} = K_{R}^{*} = K_{min} = 50$ (i.e., this corresponds to case 1) even though the optimal order quantity under profit maximization Q_{π}^{*} (= 86.60) is approximately 9.67 times larger than the optimal order quantity under return on investment maximization Q_{R}^{*} (= 8.96).

2) When (d, c) = (45, 40) or (40, 50) or (35, 60), the optimal investment decision under profit maximization is $K_{\pi}^{*} = K_{max} = 480$ while the optimal investment decision under return on investment maximization is $K_{R}^{*} = K_{min} = 50$ (i.e., this corresponds to case 4).

3) When (d, c) = (30, 70) or (25, 80) or (20, 90), the optimal investment decisions under profit maximization and return on investment maximization are identical at $K_{\pi}^{*} = K_{R}^{*} = K_{max} = 480$ (i.e., this corresponds to case 2) even though the optimal order quantities Q_{π}^{*} and Q_{R}^{*} are different.

4) When (d, c) = (15, 100), the optimal investment decision under profit maximization is $K_{\pi}^{*} = K_{min} = 50$ while the optimal investment decision under return on investment maximization is $K_{R}^{*} = K_{max} = 480$ (i.e., this corresponds to case 3).

d	c	K_π^*	Q_π^*	π_π^*	R_π^*	K_R^*	Q_R^*	R_R^*	π_R^*	Remark
50	30	50	86.60	5430.38	4.1218	50	8.96	18.5759	3411.96	$K_\pi^* = K_R^* = K_{min}$
45	40	480	15.00	4350.00	5.6154	50	9.36	11.4579	2699.10	$K_\pi^*=K_{max}$; $K_R^*=K_{min}$
40	50	480	12.65	3393.51	4.3016	50	10.01	7.0833	2101.75	$K_\pi^*=K_{max}$; $K_R^*=K_{min}$
35	60	480	10.80	2540.39	3.2000	50	10.92	4.3033	1592.17	$K_\pi^*=K_{max}$; $K_R^*=K_{min}$
30	70	480	9.26	1790.39	2.2669	480	2.37	2.9462	1650.25	$K_\pi^* = K_R^* = K_{max}$
25	80	480	7.91	1143.51	1.4756	480	2.55	1.8277	1053.52	$K_\pi^* = K_R^* = K_{max}$
20	90	480	6.67	600.00	0.8076	480	2.93	0.9321	557.11	$K_\pi^* = K_R^* = K_{max}$
15	100	50	25.98	180.39	0.2300	480	4.10	0.2574	155.83	$K_\pi^*=K_{min}$; $K_R^*=K_{max}$

Table 2. Numerical results for linear setup cost case

5) Given the other parameters fixed, as the per unit cost c increases and the sales quantity per unit time d decreases simultaneously, both the optimal profit level under profit maximization and the optimal return on investment level under return on investment maximization decrease.
6) Under the criterion of profit maximization, the higher level of profit does not necessarily represent the higher level of return on investment. For example, $\pi^*_\pi = 5430.38$ and $R^*_\pi = 4.1218$ at (d, c) = (50, 30), while $\pi^*_\pi = 4350.00$ and $R^*_\pi = 5.6154$ at (d, c) = (45, 40).
7) Under the criterion of return on investment maximization, the higher level of return on investment does not necessarily represent the higher level of profit. For example, $R^*_R = 4.3033$ and $\pi^*_R = 1592.17$ at (d, c) = (35, 60), while $R^*_R = 2.9462$ and $\pi^*_R = 1650.25$ at (d, c) = (30, 70).

HYPERBOLIC SETUP COST CASE

In this section, we consider the case that the setup cost has a hyperbolic relation to the amount of investment for setup operations. Specifically, we assume that $S(K) = \frac{\gamma}{K}$ where γ is a positive constant. For the profit maximization decision maker, the objective function and the constraints can be described as follows.

$$\underset{Q,\,K}{\text{Minimize}} \; -\pi = -pd + \frac{\gamma d}{KQ} + cd + \frac{IcQ}{2} + K$$

$$\text{s.t.} \; -K + K_{min} \leq 0$$

$$K - K_{max} \leq 0$$

For the return on investment maximization decision maker, the objective function and the constraints can be described as follows.

$$\underset{Q,\,K}{\text{Minimize}}\ -R = \frac{-pd + (\gamma d/(KQ)) + cd + (icQ/2) + K}{K + (cQ/2)}$$

$$\text{s.t. } -K + K_{min} \leq 0 \quad \text{and} \quad K - K_{max} \leq 0$$

The corresponding boundary and interior solutions which are derived from the first order necessary conditions under both profit maximization and return on investment maximization are summarized in Table 3. Moreover, in Table 3, we also present whether the second order sufficient conditions are satisfied or not at the solutions, which are obtained from the first order necessary conditions.

From Table 3, we note that, for the hyperbolic setup cost case, both the boundary solutions and the interior solutions can be optimal (cf., the linear setup cost case) no matter which optimization criterion is employed. Hence, the optimal investment decision for setup cost reduction under hyperbolic setup cost can be K_{min}, K_{max}, or K_{int}. Throughout the rest of this paper, we will denote K_{int} as the interior solution of the investment decision in the setup operations. Also, by examining the Hessian matrices of the objective functions (i.e, $-\pi$ and $-R$), it can be easily shown that the objective functions are strictly convex for the hyperbolic setup cost case. By employing both the convexity property of the objective functions and Table 3 and as by comparing the objective function values evaluated at K_{min} and K_{max}, the decision-making rules can be developed. Exhibit 1 depicts the decision-making rules for profit maximization decision maker. We note that in Exhibit 1, $Q_{\pi,min}=(\frac{2d\gamma}{K_{min}Ic})^{0.5}$ and $Q_{\pi,max}=(\frac{2d\gamma}{K_{max}Ic})^{0.5}$.

From Table 3 or Exhibit 1, the optimal interior solution of the investment

	Boundary Solutions			Interior Solutions		
	K	Q	S.O.S.C.	K	Q	S.O.N.C.
Profit Maximization	K_{min} or K_{max}	$\sqrt{\frac{2\gamma d}{KIc}}$	satisfied	$\sqrt[3]{\frac{Ic\gamma d}{2}}$	$\sqrt[3]{\frac{4\gamma d}{I^2c^2}}$	satisfied
ROI Maximization	K_{min} or K_{max}	$\frac{c+\sqrt{c^2+4B_1cK^2}}{2B_1cK}$	satisfied	$\frac{-B_2+\sqrt{B_2^2-4B_3}}{2}$	$\frac{3\gamma}{(p-c)K}$	satisfied

Table 3. The solution for hyperbolic setup cost case

Exhibit 1. The decision-making rules for hyperbolic setup cost case under profit maximization

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If $K_{min} < (\frac{Ic\gamma d}{2})^{1/3} < K_{max}$

then $K_\pi^* = (\frac{Ic\gamma d}{2})^{1/3}$ and $Q_\pi^* = (\frac{4\gamma d}{I^2c^2})^{1/3}$

else

{ if $Ic < \frac{K_{max} - K_{min}}{Q_{\pi,min} - Q_{\pi,max}}$ then $K_\pi^* = K_{min}$ and

$Q_\pi^* = (\frac{2\gamma d}{K_{min}Ic})^{1/3}$

else $K_\pi^* = K_{max}$ and $Q_\pi^* = (\frac{2\gamma d}{K_{max}Ic})^{1/3}$ }

---

decision for setup operations K_π^* increases as the inventory holding cost I, the per unit variable cost c, the sales quantity per unit time d, or the positive parameter of the setup cost function γ increases. Moreover, the optimal interior solution of the order quantity Q_π^* increase as the sales quantity per unit time d or the positive parameter of the setup cost function γ increases. On the other hand, the optimal interior solution of the order quantity Q_π^* decreases as the inventory holding cost I or the per unit variable cost c increases.

For the return on investment maximization decision maker, the decision-making rules can be described as Exhibit 2. We note that, in Exhibit 2, $Q_{R,min} = \frac{c+(c^2+4B_{1,min}cK_{min}^2)^{0.5}}{2B_{1,min}cK_{min}}$ and $Q_{R,max} = \frac{c+(c^2+4B_{1,max}cK_{max}^2)^{0.5}}{2B_{1,min}cK_{min}}$.

Exhibit 2. The decision-making rules for hyperbolic setup cost case under return on investment maximization

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Let $B_{1,min} = \dfrac{pd-cd-K_{min}+iK_{min}}{2\gamma d}$, $B_{1,max} = \dfrac{pd-cd-K_{max}+iK_{max}}{2\gamma d}$,

$B_2 = \dfrac{9\gamma c(1-i)}{2d(p-c)^2}$, and $B_3 = \dfrac{-3c\gamma}{2(p-c)}$.

If $K_{min} < \dfrac{-B_2 + (B_2^2 - 4B_3)^{0.5}}{2} < K_{max}$

then $K_R^* = \dfrac{-B_2 + (B_2^2 - 4B_3)^{0.5}}{2}$ and $Q_R^* = \dfrac{3\gamma}{(p-c)K_R^*}$.

Else

{ if $\dfrac{K_{max}}{Q_{R,min}^2} > \dfrac{K_{min}}{Q_{R,max}^2}$, then $K_R^* = K_{min}$ and $Q_R^* = Q_{R,min}$

else $K_R^* = K_{max}$ and $Q_R^* = Q_{R,max}$ }

From Exhibit 1 and Exhibit 2, we note that the combination of the optimal investment decisions in the setup operations under profit maximization and return on investment maximization are among the following nine cases: 1) $K_\pi^* = K_R^* = K_{min}$, 2) $K_\pi^* = K_{min}$; $K_R^* = K_{max}$, 3) $K_\pi^* = K_{min}$; $K_R^* = K_{int}$, 4) $K_\pi^* = K_{int}$; $K_R^* = K_{min}$, 5) $K_\pi^* = K_R^* = K_{int}$, 6) $K_\pi^* = K_{int}$; $K_R^* = K_{max}$, 7) $K_\pi^* = K_{max}$; $K_R^* = K_{min}$, 8) $K_\pi^* = K_{max}$; $K_R^* = K_{int}$, and 9) $K_\pi^* = K_R^* = K_{max}$.

In order to illustrate the features of the hyperbolic setup cost case under both profit maximization and return on investment maximization, we provide the following numerical example.

Example 2. Suppose $\gamma = 15000$, $d = 25$, $I = 0.2$, $i = 0.1$, $c = 100$, $p = 150$.

We plot the negative profit $(-\pi)$ surface and the negative return on investment $(-R)$ surface in Figure 3 and Figure 4, which show that the interior solutions derived from solving the first order necessary conditions (i.e., $(K_\pi, Q_\pi) = (155.36, 15.54)$ and $(K_R, Q_R) = (169.03, 5.32)$) are global minimum solutions (this is due to the convexity of the objective functions).

We now demonstrate that all of those nine cases can exist by only changing the per unit cost c and the lower and upper limits of the amount of investment for setup operations (i.e., K_{min} and K_{max}). The results are presented in Table 4.

By examining Table 4 carefully, we make the following interesting observations.

1) The optimal investment decisions in the setup operations may be identical no matter which optimization criterion is employed. For example, when $(c, K_{min}, K_{max}) = (60, 90, 110)$, $K_\pi^* = K_R^* = K_{max}$; or when $(c, K_{min}, K_{max}) = (100, 170, 190)$, $K_\pi^* = K_R^* = K_{min}$.
2) The optimal investment decisions in the setup operations may be in opposite direction in the sense that $K_\pi^* = K_{min}$; $K_R^* = K_{max}$ or $K_\pi^* = K_{max}$; $K_R^* = K_{min}$. For example, when $(c, K_{min}, K_{max}) = (40, 90, 110)$, $K_\pi^* = K_{max}$ and $K_R^* = K_{min}$ or when $(c, K_{min}, K_{max}) = (120, 168, 175)$, $K_\pi^* = K_{min}$; $K_R^* = K_{max}$.
3) The optimal investment decisions in the setup operations may occur at interior points for both profit maximization and return on investment maximization. For example, when $(c, K_{min}, K_{max}) = (80, 130, 150)$, K_π^*

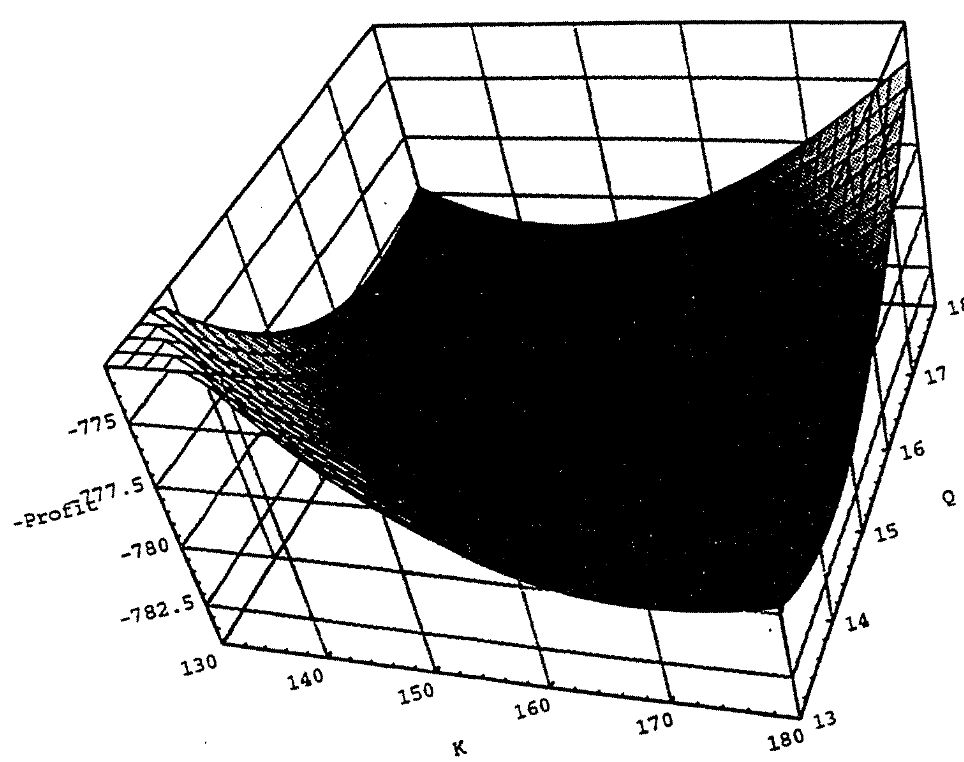

Figure 3. Negative profit $(-\pi)$ surface with respect to Q and K under hyperbolic setup cost function

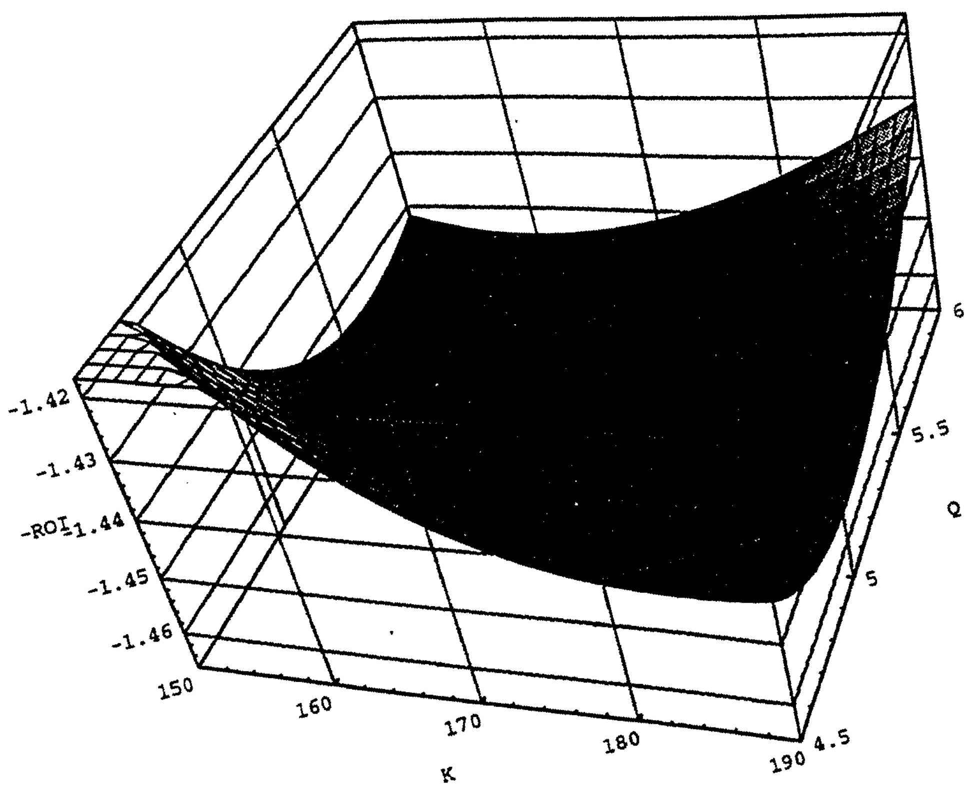

Figure 4. Negative return on investment (-R) surface with respect to Q and K under hyperbolic setup cost function

c	K_{min}	K_{max}	K^*_π	Q^*_π	π^*_π	R^*_π	K^*_R	Q^*_R	R^*_R	π^*_R	Remark
40	90	110	110	29.19	2406.45	3.5527	90	4.64	9.5883	1743.45	$K^*_\pi=K_{max}$; $K^*_R=K_{min}$
50	90	120	120	25.00	2130.00	2.9430	123.31	4.49	7.3196	1543.54	$K^*_\pi=K_{max}$; $K^*_R=K_{int}$
60	90	110	110	23.84	1853.96	2.3334	110	4.47	5.5876	1350.52	$K^*_\pi = K^*_R = K_{max}$
70	130	150	137.95	19.71	1586.16	1.9995	130	4.37	4.2220	1179.32	$K^*_\pi=K_{int}$; $K^*_R=K_{min}$
80	130	150	144.23	18.02	1317.33	1.6062	141.74	4.54	3.1154	989.19	$K^*_\pi = K^*_R = K_{int}$
90	130	152	150.00	16.67	1050.00	1.2498	152	4.87	2.2080	797.58	$K^*_\pi=K_{int}$; $K^*_R=K_{max}$
100	170	190	170	14.85	783.92	0.9394	170	5.31	1.4651	611.48	$K^*_\pi = K^*_R = K_{min}$
110	170	190	170	14.16	518.87	0.6285	160.83	6.29	0.8636	398.70	$K^*_\pi=K_{min}$; $K^*_R=K_{int}$
120	168	175	168	13.64	254.71	0.3412	175	8.48	0.3970	220.54	$K^*_\pi=K_{min}$; $K^*_R=K_{max}$

Table 4. Numerical results for hyperbolic setup cost case

$= 144.23$ and $K_R^* = 141.74$. In this case, both K_π^* and K_R^* are interior points.

4) The optimal investment decision under profit maximization may occur at boundary points (interior points) while the optimal investment decision under return on investment maximization may occur at interior points (boundary points). For example, when $(c, K_{min}, K_{max}) = (50, 90, 120)$, $K_\pi^* = K_{max}$ and $K_R^* = K_{int}$; or when $(c, K_{min}, K_{max}) = (110, 170, 190)$, $K_\pi^* = K_{min}$ and $K_R^* = K_{int}$; or when $(c, K_{min}, K_{max}) = (70, 130, 150)$, $K_\pi^* = K_{int}$ and $K_R^* = K_{min}$; or when $(c, K_{min}, K_{max}) = (90, 130, 152)$, $K_\pi^* = K_{int}$ and $K_R^* = K_{max}$.

By comparing the optimal investment decisions for linear setup cost case and the optimal investment decisions for hyperbolic setup cost case, it can be easily observed that the optimal investment decisions for linear setup cost case is more sensitive than the optimal investment decisions for hyperbolic setup cost case. This is because that the optimal investment decisions for linear setup cost case occurs only at boundary points (i.e., K_{max} or K_{min}) while the optimal investment decisions for hyperbolic setup cost case may occur at both boundary points and interior points (i.e., K_{max}, K_{min}, or K_{int}).

CONCLUDING REMARKS

In this paper, we constructed and analyzed inventory and investment in setup cost operations policies under profit maximization and return on investment maximization for the decision maker of the inventory system. First, we showed how inventory and investment in setup operations policies under profit maximization and return on investment maximization can be formulated for general functional form of the investment in setup operations. From these formulations, the optimality conditions and the corresponding economic interpretations are obtained. Next, for the specific cases of the linear setup cost and the hyperbolic setup cost, the optimal closed-form solutions are obtained and several interesting managerial insights are presented.

The models developed in this paper relates general practices since numerous industries and firms apply EOQ based decision making for their inventory systems. There are several possible extensions that will further improve the relevance of our models to general practices. They include incorporation of more sophisticated features such as shortages, delivery lags, and stochastic demand rates, etc. From the perspective of investing in setup operations, it would be of interest to study the allocation of the investment in setup operations. For example, how much should be invested in purchasing or leasing new equipments and how much should be invested in labor's training and wages, etc. From the perspective of optimization criterion, it would be of interest to study the effects of investing in setup operations on process quality improvement, effective capacity and flexibility improvement (see e.g., Porteus [11], and Spence and Porteus [15]) in conjunction with the optimization criterion of return on investment.

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CHAPTER IV.

A MULTI-PRODUCT EOQ MODEL WITH PRICING CONSIDERATION -T. C. E. CHENG'S MODEL REVISITED

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ABSTRACT

We present two major revisions/corrections regarding a recent paper by T. C. E. Cheng [1]. First, we note that a critical assumption of the equal replenishment cycle length for all products is stated, but not incorporated into the mathematical formulation in Cheng [1]. In this paper, we re-formulate the problem with the equal replenishment cycle length incorporated and derive the corresponding Kuhn-Tucker optimality conditions. Next, under the linear demand assumption, we show that the closed-form solutions provided by Cheng [1] may result in non-optimal solutions. The reason is that Cheng [1] failed to derive conditions under which the closed-form solutions may be optimal. In this paper, by employing the trigonometric methods (see e.g., Porteus [7]), we derive the optimal closed-form solution that is unique and obtain the conditions under which the optimal closed-form solution is valid.

INTRODUCTION

In a recent paper, Cheng [1] proposes a multi-product EOQ model that integrates the pricing and order sizing decisions to maximize profit with storage space and inventory investment constraints. This EOQ problem is formulated as a constrained non-linear optimization problem and the corresponding Kuhn-Tucker conditions are derived for the optimal solutions.

Even though Cheng [1] makes a valuable contribution in integrating inventory and pricing policies, we believe that the EOQ model needs two major revisions/ corrections - one in the model formulation phase and the other in the closed-form solution derivation phase under the linear demand assumption.

In Cheng's paper [1], a critical and simplifying assumption is made that all products have equal replenishment cycle length. Under this assumption, he formulates the multi-product EOQ problem as a nonlinear programming, which maximizes profit over the demand rate and the order size for each product by considering the storage space and inventory investment constraints. In the formulation, however, the equal replenishment cycle length assumption is not mathematically incorporated. Therefore, the optimal solutions from the model may result in unequal replenishment cycle lengths for some products, violating the stated assumption. In this paper, we will explicitly incorporate the equal replenishment cycle length assumption in the formulation. Consequently, we obtain the Kuhn-Tucker optimality conditions that are substantially different from those shown in Cheng [1].

Also, under the linear demand assumption, Cheng [1] provides optimal closed-form solutions. We will show that the closed-form solutions, however, may result in non-optimal solutions via numerical examples. The reason is that Cheng [1] failed to derive conditions under which the closed-form solutions may be optimal. In this paper, by employing the trigonometric methods (see e.g., Chapter 2 of Griffiths [6]), we derive the optimal closed-form solution that is unique and obtain the conditions under which the optimal closed-form solution is valid.

BASIC MODEL

In order to formulate the basic model, as in Cheng's paper [1], the following definitions and assumptions are employed.

n = total number of products produced by the firm;

Q_i = demand rate for product i;

$\underset{\sim}{Q} = (Q_1, Q_2, Q_3, \cdots, Q_n)$, the demand rate vector;

q_i = order size of product i;

$\underset{\sim}{q} = (q_1, q_2, q_3, \cdots, q_n)$, the order size vector;

s_i = order cost per batch of product i;

r_i = unit cost of production of product i;

j = fractional inventory carrying cost rate;

T = length of a replenishment cycle;

f_i = storage space requirement per unit of product i;

P_i = unit selling price of product i;

$\Pi(\underset{\sim}{Q}, \underset{\sim}{q}, T)$ = total profit derived from the sale of the products;

F = total fixed cost of production and administration;

A = total storage space available;

I = maximum inventory investment allowable.

The following basic assumptions about the model are made:

(A1) All products have the equal replenishment cycle length T.

i.e., $T = \frac{q_i}{Q_i}$, $i = 1, \cdots, n$.

(A2) Replenishments of the products are instantaneous.

(A3) No backorder is permitted.

(A4) The demand rates are uniform and continuous.

(A5) The demand functions of the products are given as follows:

$$P_i = h_i(Q_i),\ 1 \leq i \leq n$$

where $h_i(\cdot)$ is a function of Q_i which, in general, is monotonically decreasing.

In addition, in this paper as well as implicitly in Cheng [1], it is assumed that all products are replenished at the same time.

Under these definitions and assumptions, the revenue per cycle is $\sum_{i=1}^{n} P_i q_i$, the total production cost per cycle plus the total fixed cost of production and administration per cycle are $\sum_{i=1}^{n} (s_i + r_i q_i + \frac{j r_i q_i^2}{2Q_i}) + FT$. Therefore, the profit per cycle, which is the revenue less the cost, is given by $\sum_{i=1}^{n} P_i q_i - \sum_{i=1}^{n} (s_i + r_i q_i + \frac{j r_i q_i^2}{2Q_i}) - FT$. The corresponding profit per unit time can be obtained by dividing the profit per cycle by the cycle length $\frac{q_i}{Q_i}$. The objective of our model is to maximize the profit per unit time subject to the storage and inventory investment constraints as well as the equal replenishment cycle length restriction. Namely,

$$\text{Maximize } \Pi(\underset{\sim}{Q}, \underset{\sim}{q}, T) = \sum_{i=1}^{n} \{h_i(Q_i)Q_i - \frac{s_iQ_i}{q_i} - r_iQ_i - \frac{jr_iq_i}{2}\} - F \quad (1)$$

subject to :

1. $$\sum_{i=1}^{n} f_iq_i \leq A \quad (2)$$

2. $$\sum_{i=1}^{n} \frac{jr_iq_i^2}{2Q_i} \leq I \quad (3)$$

3. $$T = \frac{q_i}{Q_i} \text{ for } i = 1, 2, 3, 4, \cdots\cdots, n \quad (4)$$

where q_i, Q_i are non-negative for $1 \leq i \leq n$.

In particular, the third constraint explicitly expresses the critical assumption that all products have the equal replenishment cycle. In such a case, the decision variables are not only the demand rate and the order size for each product but also the equal replenishment cycle length. We note that, in Cheng's paper [1], the third constraint is not incorporated in the formulation even though the assumption is stated in the problem. This omission, we believe, is significant in that the optimization of the formulation in Cheng [1] may result in infeasible solutions. To emphasize the differences between Cheng [1] formulation and our revised formulation, a numerical example is provided at the end of this section.

Also, in order to meet the standard form of nonlinear programming (see, e.g., Chiang [3] or Intriligator [4]), we will employ an equivalent set of constraints for the third constraint as follows.

$$T \leq \frac{q_i}{Q_i} \text{ and } T \geq \frac{q_i}{Q_i}. \quad (5)$$

Given the revised formulation of the problem (1)-(3) and (5), we have the following Lagrangian function:

$$L = \sum_{i=1}^{n} \{h_i(Q_i)Q_i - \frac{s_iQ_i}{q_i} - r_iQ_i - \frac{jr_iq_i}{2}\} - F + \lambda_1\{A - \sum_{i=1}^{n} f_iq_i\} +$$

$$\lambda_2\{I - \sum_{i=1}^{n} \frac{jr_iq_i^2}{2Q_i}\} + \sum_{i=1}^{n} \mu_i\{-T + \frac{q_i}{Q_i}\} + \sum_{i=1}^{n} \omega_i\{ -\frac{q_i}{Q_i} + T\} \qquad (6)$$

where λ_1, λ_2, μ_i, and ω_i for $i = 1, \cdots, n$, are the Lagrangian multipliers.

Next, invoking the Kuhn-Tucker optimality conditions for the Lagrangian function (6), we obtain the necessary conditions for the optimal solution as follows.

Condition 1

$$Q_i\frac{\partial L}{\partial Q_i} = 0, \ 1 \le i \le n.$$

or

$$Q_i \{ h_i(Q_i) + Q_i\frac{\partial h_i(Q_i)}{\partial Q_i} - \frac{s_i}{q_i} - r_i + \lambda_2\frac{jr_iq_i^2}{2Q_i^2} - \mu_i\frac{q_i}{Q_i^2} + \omega_i\frac{q_i}{Q_i^2} \} = 0 \qquad (7)$$

Condition 2

$$\frac{\partial L}{\partial Q_i} \le 0, \ 1 \le i \le n.$$

or

$$h_i(Q_i) + Q_i\frac{\partial h_i(Q_i)}{\partial Q_i} - \frac{s_i}{q_i} - r_i + \lambda_2\frac{jr_iq_i^2}{2Q_i^2} - \mu_i\frac{q_i}{Q_i^2} + \omega_i\frac{q_i}{Q_i^2} \le 0 \qquad (8)$$

Condition 3

$$q_i \frac{\partial L}{\partial q_i} = 0, \ 1 \leq i \leq n.$$

or

$$q_i \{ \frac{s_i Q_i}{q_i^2} - \frac{jr_i}{2} - \lambda_1 f_i - \lambda_2 \frac{jr_i q_i}{Q_i} + \mu_i \frac{1}{Q_i} - \omega_i \frac{1}{Q_i} \} = 0 \qquad (9)$$

Condition 4

$$\frac{\partial L}{\partial q_i} \leq 0, \ 1 \leq i \leq n.$$

or

$$\frac{s_i Q_i}{q_i^2} - \frac{jr_i}{2} - \lambda_1 f_i - \lambda_2 \frac{jr_i q_i}{Q_i} + \mu_i \frac{1}{Q_i} - \omega_i \frac{1}{Q_i} \leq 0 \qquad (10)$$

Condition 5

$$\lambda_1 \frac{\partial L}{\partial \lambda_1} = 0$$

or

$$\lambda_1 \{ A - \sum_{i=1}^{n} f_i q_i \} = 0 \qquad (11)$$

Condition 6

$$\frac{\partial L}{\partial \lambda_1} \geq 0$$

or

$$A - \sum_{i=1}^{n} f_i q_i \geq 0 \qquad (12)$$

Condition 7

$$\lambda_2 \frac{\partial L}{\partial \lambda_2} = 0$$

or

$$\lambda_2 \{ I - \sum_{i=1}^{n} \frac{jr_i q_i^2}{2Q_i} \} = 0 \qquad (13)$$

Condition 8

$$\frac{\partial L}{\partial \lambda_2} \geq 0$$

or

$$I - \sum_{i=1}^{n} \frac{jr_i q_i^2}{2Q_i} \geq 0 \tag{14}$$

Condition 9

$$T \frac{\partial L}{\partial T} = 0$$

or

$$T\{\sum_{i=1}^{n} (-\mu_i + \omega_i)\} = 0 \tag{15}$$

Condition 10

$$\frac{\partial L}{\partial T} \leq 0$$

or

$$\sum_{i=1}^{n} (-\mu_i + \omega_i) \leq 0 \tag{16}$$

Condition 11

$$\mu_i \frac{\partial L}{\partial \mu_i} = 0, \ 1 \leq i \leq n.$$

or

$$\mu_i\{-T + \frac{q_i}{Q_i}\} = 0 \tag{17}$$

Condition 12

$$\frac{\partial L}{\partial \mu_i} \geq 0, \ 1 \leq i \leq n.$$

or

$$-T + \frac{q_i}{Q_i} \geq 0 \tag{18}$$

Condition 13

$$\omega_i \frac{\partial L}{\partial \omega_i} = 0, \ 1 \le i \le n.$$

or

$$\omega_i \{ -\frac{q_i}{Q_i} + T\} = 0 \tag{19}$$

Condition 14

$$\frac{\partial L}{\partial \omega_i} \ge 0, \ 1 \le i \le n,$$

or

$$-\frac{q_i}{Q_i} + T \ge 0 \tag{20}$$

Condition 15

$$Q_i, \ q_i, \ T, \ \lambda_1, \ \lambda_2, \ \mu_i, \ \omega_i \ge 0, \ 1 \le i \le n. \tag{21}$$

We note that the above conditions (7) — (21) are only the necessary conditions for optimality. We will refer the readers to optimization textbooks such as Luenberger [5] regarding the second order sufficient conditions (SOSC) for the optimality. We also note that there are several widely-used commercial software packages (such as GINO) which efficiently computes the optimal solutions for non-linear optimization problem. Finally, we note that there is an implicit assumption that the profit level at the optimality for each product i (excluding the total fixed cost F, which is independent of i) is non-negative. The reason is that few firms will operate with negative profit for any product in the long run.

So far, we have incorporated the condition of equal replenishment cycle length into the mathematical formulation and obtained the corresponding Kuhn-Tucker conditions. In order to illustrate the differences between Cheng's formulation and our revised formulation, we construct the following numerical example:

$j = 0.4 \qquad A = 500 \qquad I = 50 \qquad F = 0$

$P_1 = h_1(Q_1) = \alpha_1 - \beta_1 \ln(Q_1); \quad P_2 = h_2(Q_2) = \alpha_2 - \beta_2 \ln(Q_2)$

$\alpha_1 = 100 \qquad \alpha_2 = 120 \qquad \beta_1 = 20 \qquad \beta_2 = 15$

$s_1 = 18 \quad s_2 = 30 \quad r_1 = 1.2 \quad r_2 = 1.8 \quad f_1 = 0.9 \quad f_2 = 1.5$

Table 1. Comparison Between Cheng's Formulationand Revised Formulation

		Cheng's Formulation	Revised Formulation
demand rate	Q_1^*	50.68	51.18
	Q_2^*	965.91	963.36
order size	q_1^*	61.78	17.16
	q_2^*	284.43	323.04
cycle length	T_1^*	1.2192	0.3353
	T_2^*	0.2495	0.3353
storage space constraint		non-binding 482.25 < 500	binding 500 = 500
inventory investment constraint		non-binding 48.23 < 50	non-binding 40.38 < 50
cycle length constraint		violated, $T_1 \neq T_2$	satisfied, $T_1 = T_2$
feasibility		infeasible	feasible and optimal

By employing an optimization software package GINO and an IBM PC 386, we easily obtain the solutions for both formulations. The corresponding results are summarized in Table 1.

From Table 1, we observe the following:

1) The optimal values of decision variables can be quite different. This

implies that a manager might end up with an infeasible decision if he had followed Cheng [1].

2) The condition of equal replenishment cycle length may have an impact on the storage space constraint or the inventory investment constraint. For example, the storage space constraint is non-binding in Cheng's Formulation while it is binding in Revised Formulation.

As shown in 1) and 2), as well as in Table 1, not incorporating the condition of equal replenishment cycle length may substantially distort the optimal solutions, and the managerial consequence (due to infeasibility, etc.) may be substantial.

LINEAR DEMAND FUNCTION

In Cheng [1], under the linear demand assumption, closed-form solutions are derived through relations (21) - (28) relying on Standard Mathematical Table [2]. First of all, we believe that there is a typographical error in equation (27). Instead of $B_i = -(-(b_i/2) + ((b_i^2/4) + (a_i^3/27))^{0.5})^{1/3}$ as in Cheng [1], $B_i = (-(b_i/2) - ((b_i^2/4) + (a_i^3/27))^{0.5})^{1/3}$.

Even if equation (27) of Cheng [1] were correct, a serious problem arises in equations (28) and subsequent sentences in page 534 of Cheng [1]. Equations (28) provide three candidates for the closed-form solution. On line 5 and 6 in page 534, the following is stated verbatim: "It follows that $Q_i^* = y_i^2$ and, from equation (21), $q_i^* = y_i(2s_i/jr_i)^{1/2}$. Finally, we have to determine whether the solutions Q^* and q^* satisfy the constraints. If they do, then they are optimal."

The problem with the above statement is that even if such solutions Q^* and q^* were found, they may not be optimal solutions! The reason is that Cheng [1] failed to derive conditions under which equations (28) lead to the closed-form optimal solutions. A related problem is that Cheng [1] also failed to present a mechanism to determine the true optimal candidate out of the three candidates of equations (28). We demonstrate these problems by way of the following numerical example as well as the example at the end of this section.

Let us suppose that $s_k = 200$, $j = 0.5$, $r_k = 40$, $m_k = 10$, and $P_k^0 = 100$ for product k. Then, equations (24)-(28) in Cheng [1] lead to $Q_k = 4.10$, $0.95-0.565i$, or $0.95+0.565i$. Obviously, the imaginary numbers are unrealistic to be the optimal solutions. In the case of $Q_k = 4.10$, the corresponding q_k is equal to -9.0598, which is also unrealistic.

As we can see from this numerical example, clarification and improvement are necessary. In order to do so, we employ the trigonometric methods (see e.g., Chapter 2 of Griffiths [6] or appendix of Porteus [7]), and derive the optimal closed-form solution that is unique and obtain the conditions under which the optimal closed-form solution is valid. Specific derivations are as follows.

As in Cheng [1], we assume that there is a linear relationship between the unit selling price and demand rate for the products. Specifically, we denote the price-demand function as follows.

$$P_i = h_i(Q_i) = P_i^0 - m_iQ_i, \ 1 \le i \le n. \tag{22}$$

where P_i^0, $m_i > 0$ are arbitrary constants and $P_i^0 > m_iQ_i$.

Following the solving procedure in Cheng's paper [1], we assume for the time being that constraint sets (2) and (3) are inactive while constraint set (4) is satisfied. Hence, we can obtain the following necessary conditions for the optimal solutions.

$$\frac{\partial \Pi}{\partial Q_i} = P_i^0 - 2m_iQ_i - r_i - \frac{s_i}{q_i} = 0 \tag{23}$$

$$\frac{\partial \Pi}{\partial q_i} = \frac{s_iQ_i}{q_i^2} - \frac{jr_i}{2} = 0 \tag{24}$$

By substituting and rearranging the relation $q_i = (\frac{2s_iQ_i}{jr_i})^{0.5}$ from (24) into (23), we obtain the optimality condition for Q_i as follows:

$$Q_i^{1.5} + (\frac{r_i - P_i^0}{2m_i})Q_i^{0.5} + (\frac{jr_is_i}{8m_i^2})^{0.5} = 0 \tag{25}$$

By employing the trigonometric methods (see e.g., Chapter 2 of Griffiths [6] or appendix of Porteus [7]), we obtain the optimal demand rate, Q_i, as follows.

$$Q_i^* = \frac{2(P_i^0 - r_i)}{3m_i} \cos^2(\frac{\theta_i}{3}) \tag{26}$$

$$\text{where } \cos\theta_i = -(\frac{27m_ijr_is_i}{4(P_i^0 - r_i)^3})^{0.5}, \text{ and } \frac{\pi}{2} < \theta \leq \frac{3\pi}{4}.$$

We note that the upper bound of $3\pi/4$ on the critical angle θ is obtained from the assumption that the resulting profit for each product i is non-negative. On the other hand, the lower bound of $\pi/2$ on the critical angle θ implies that parameters m_i, j, r_i, and s_i should all be strictly positive in order for the profit maximization EOQ model to be non-degenerate. From (24), the corresponding order size q_i is:

$$q_i^* = (\frac{4s_i(P_i^0 - r_i)}{3m_ijr_i})^{0.5} \cos(\frac{\theta_i}{3}) \tag{27}$$

for $\frac{\pi}{2} < \theta \leq \frac{3\pi}{4}$ and $\cos\theta_i = -\left(\frac{27m_i j r_i s_i}{4(P_i^0 - r_i)^3}\right)^{0.5}$.

For $\frac{\pi}{2} < \theta \leq \frac{3\pi}{4}$, it can be easily verified that the second order sufficient conditions for the profit maximization are satisfied at (Q_i, q_i) given by expression (26) and (27).

From (26) and (27), we obtain the corresponding optimal price P_i and replenishment cycle length T_i as follows.

For $\frac{\pi}{2} < \theta \leq \frac{3\pi}{4}$ and $\cos\theta_i = -\left(\frac{27m_i j r_i s_i}{4(P_i^0 - r_i)^3}\right)^{0.5}$,

$$P_i^* = P_i^0 - \frac{2(P_i^0 - r_i)}{3}\cos^2\left(\frac{\theta_i}{3}\right) \tag{28}$$

$$T_i^* = \left(\frac{3s_i m_i}{j r_i (P_i^0 - r_i)}\right)^{0.5}\left(\cos\left(\frac{\theta_i}{3}\right)\right)^{-1} \tag{29}$$

Finally, we have to examine whether the solution $\underset{\sim}{Q}^*$, $\underset{\sim}{q}^*$ satisfy the constraint sets (2), (3), and (4). If they do, then they are optimal. If they do not, then the constraints are active and we have to employ the Lagrangian function and the Kuhn-Tucker conditions, as discussed in the previous section, to find the optimal solutions.

In order to illustrate some of the features discussed in this section, we will solve a two-product profit maximization problem over the demand rate Q_i and order size q_i for i = 1, 2. We will solve by the trigonometric methods first. Let us assume that the following parametric values are provided.

$j = 0.5$	$A = 60$	$I = 25$
$P_1^0 = 100$	$P_2^0 = 67.49$	
$m_1 = 11.06$	$m_2 = 12$	

$s_1 = 10$ $\qquad$ $s_2 = 12$

$r_1 = 1$ $\qquad$ $r_2 = 2$

$f_1 = 2$ $\qquad$ $f_2 = 3$

From equations (26) and (27), the optimal demand rate and order size for product 1 and 2 are given by

$$Q_1^* = 4.44, \qquad q_1^* = 13.33;$$
$$Q_2^* = 2.67, \qquad q_2^* = 8.00.$$

It can be easily verified that the corresponding replenishment cycle lengths for product 1 and product 2 are both equal to 3. On the other hand, the corresponding storage space and inventory investment in this problem are 50.67 and 22. Both the storage space and inventory investment constraints are satisfied at the optimal solution.

In contrast to the trigonometric methods, if equations (28) of Cheng [1] are used, the demand rate for product 1 will be 0.000255, 4.50923, or 4.44167. We note that no mechanism is provided in Cheng [1] that will determine which one among the three candidates is the optimal solution. cf. the trigonometric methods result in the unique demand rate.

SUMMARY

In this paper, we have presented two major revisions/ corrections regarding a recent paper by T. C. E. Cheng [1]— "An EOQ Model with Pricing Consideration". First, we pointed out that the critical assumption of equal replenishment cycle length for each product was not incorporated into his model formulation. The correct model should have contained n+2 constraints instead of two, and the number of the optimality Kuhn-Tucker conditions should have been fifteen instead of nine. Next, in the case of linear demand functions, we indicated that the solutions provided by Cheng [1] may result in non-optimal solution, or multiple candidates. By employing the trigonometric methods, we derived the optimal closed-form solution that is unique and obtained the conditions under which the optimal closed-form solution is valid.

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CHAPTER V.

A TWO-STAGE BROKERAGE SYSTEM FOR ELECTRIC POWER TRANSACTIONS

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Abstract

We propose a two-stage brokerage system for electric power transactions. By employing economic analysis and linear programming at each stage, we show that significant gains in economic efficiency can be achieved.

Key Words: Costing, Brokerage, Linear Programming, Economic Efficiency, Electric Power, Interconnected Power Network.

1. Introduction

The electric power industry in the United States is currently facing a drastic transformation from a traditional, regulated, and vertically integrated environment to a de-regulated and competitive environment [6]. A primary motivation for this transformation is to improve the economic efficiency in the power industry. A critical research area where the power industry can improve the economic efficiency is that of power interchange in an interconnected power system. The power interchange may improve the economic efficiency because there exist some potential savings whenever the difference in incremental production costs among utilities is significant and excess production capacities exist. In this paper, for the power interchange transactions among utilities, we present a two-stage brokerage system that will result in significant gains in economic efficiency.

The purpose of a brokerage system is to maximize the total benefit (saving) by matching the bids from buyers and sellers. The conventional brokerage system (see Doty and McEntire [1] or Fahd, Richards, and Sheble [3] for details) matches the highest purchase bid with the lowest sale bid, the second highest purchase bid with the second lowest sale bid, and so on. The matching process terminates when a viable match no longer exists. Doty and McEntire [1] proposed two algorithms to improve the conventional brokerage systems: one employed a network flow algorithm and the other utilized dynamic programming techniques. Fahd, Richards, and Sheble [3] implemented an energy brokerage system by employing linear programming. In their model, buyers and sellers can use the transmission networks of intermediate utilities and pay for the transmission service charges. The transmission service charges of their model are assumed to be strictly positive. This assumption, however, is not universally accepted. That is, the transmission service charges may be positive or negative (see e.g., Li and David [4]). This implies that the transmission service charges, in their model, do not reflect the true cost/benefit of the intermediate transmission utilities. In contrast, in this paper, we attempt to design the transmission service charges so as to accurately reflect the true cost/benefit of the intermediate transmission utilities by considering physical aspects of the transmission such as the prevailing direction of power flow. Furthermore, in our paper, the sellers, buyers, and the intermediate transmission utilities actively calculate their net costs and benefits in determining the sale and purchase bids and the transmission service charges. This feature differentiates our paper from the extant literature on electric power brokerage systems.

In our model, at the first stage, the brokerage system will match the bids from buyers and sellers. At this stage, the brokerage system does not take the transmission service charges of the intermediate transmission utilities into account. At the second stage, based on the matching bids of buyers and sellers, the brokerage system determines the route(s) with the minimum transmission service charges. The transmission service charges are based on an economic dispatch calculation employing a transportation method (see e.g., Lee, Thorne, and Hill. [5]). The following assumptions are made for the model:

1) Intermediate transmission utilities are neither buyers nor sellers.

2) The transmission service charges are small relative to the total savings from power interchange transactions.
3) The electric power flow can be treated as a commodity that can be transported by any selected transmission route subject to capacity restrictions (with advanced transmission systems such as the flexible AC transmission system (FACTS), it is a reasonable assumption, see e.g., Li and David [4]).

The rest of this paper is organized as follows. First, we briefly review an economic dispatch model employing a transportation method. Then, we will show how the two-stage brokerage system is constructed. Also, in order to elucidate the two-stage brokerage system, several numerical examples are provided. Finally, the concluding remarks are made.

2. Review of a Transportation-Type Economic Dispatch Model

The conventional economic dispatch [8] concerns with the minimization of production cost subject to demand-supply relations and generation capacity constraints for an electric utility. The optimal solutions, however, do not specify the power flow direction on each transmission line. In order to rectify this shortcoming and specify the power flow directions, Lee, Thorne, and Hill [5] proposed an alternative economic dispatch model employing a transportation method. Our model will utilize this transportation-type economic dispatch model in determining bids for buyers and sellers as well as transmission service charges for intermediate transmission utilities. Hence, we first briefly review the transportation-type economic dispatch model. We employ the same notations as in Lee, Thorne, and Hill [5] for an I-generator, N-bus, M-line system.

$I(n)$ = number of generators connected to bus n,

$M(n)$ = number of lines connected to bus n,

G_i = MW produced at generator i,

$F_i(G_i)$= the production cost for Gi MW at generator i,

D_n = MW load at bus n,

R_m = the resistance of transmission line m, measured in 1/MW,

T_m = MW transmitted on line m.

The subscripts i, n, m are dummy counters for I, N, M respectively. Also, in Lee, Thorne, and Hill [5], the transmission loss is directly related to the amount of power on a transmission line and can be approximately expressed by the following relation (see Elgerd [2] for details).

$$P_{Lm} = R_m T_m^2 \tag{1}$$

where P_{Lm} is the transmission loss on line m.

Therefore, the total transmission loss in the system, $P_L = \sum^{M} R_m T_m^2$. Now, if we denote the marginal cost for transmission loss by h, then the cost function for transmission loss in the system will be hP_L.

Under these definitions and assumptions, the economic dispatch problem can be mathematically formulated as follows.

$$\text{Minimize: } F = \sum^{I} F_i(G_i) + hP_L \tag{2}$$

$$\text{subject to : } \sum^{I(n)} G_i - D_n + \sum^{M(n)} T_m = 0 \tag{3}$$

$$\underline{G_i} \leq G_i \leq \overline{G_i} \quad \text{for all i,} \tag{4}$$

$$0 \leq |T_m| \leq \overline{T_m} \quad \text{for all m.} \tag{5}$$

It is noted that the decision variables in the above system are G_i's and

T_m's. Equation (3) represents the law of conservation at each bus (i.e., flow into the bus = flow out of the bus). Bar under and bar over represent lower and upper limits on the decision variables. Also, in this paper, we assume that the production cost function $F_i(G_i)$ has a linear relation with respect to the generation output G_i (see e.g., Fahd, Richards, and Sheble [3] or Wood & Wollenberg [8]). We will first introduce the following example to illustrate the transportation-type economic dispatch model. Also, this example will be further utilized throughout this paper to illustrate the features of the two-stage brokerage system for electric power transactions.

	G_1	G_2	G_3	T_1	T_2	T_3	T_4	Cost
Utility 1	100	200	-	12.5	37.5	62.5	-	5303.1
Utility 2	150	350	200	27.3	77.3	272.7	150	19018.2
Utility 3	350	150	-	92.9	257.1	207.1	-	9988.6
Utility 4	250	200	150	60.0	210.0	90.0	90	7114.0
Utility 5	250	150	150	116.7	16.7	150.0	150	14916.7

Table 1. Optimal solutions by employing the transportation-type economic dispatch method

<u>Example 1</u> We now consider the five-utility interconnected electric power system shown in Figure 1. The relevant information of the generators and the transmission lines for each utility is shown in Appendix.

The optimal solution for each utility by employing the transportation-type economic dispatch is shown in Table 1.

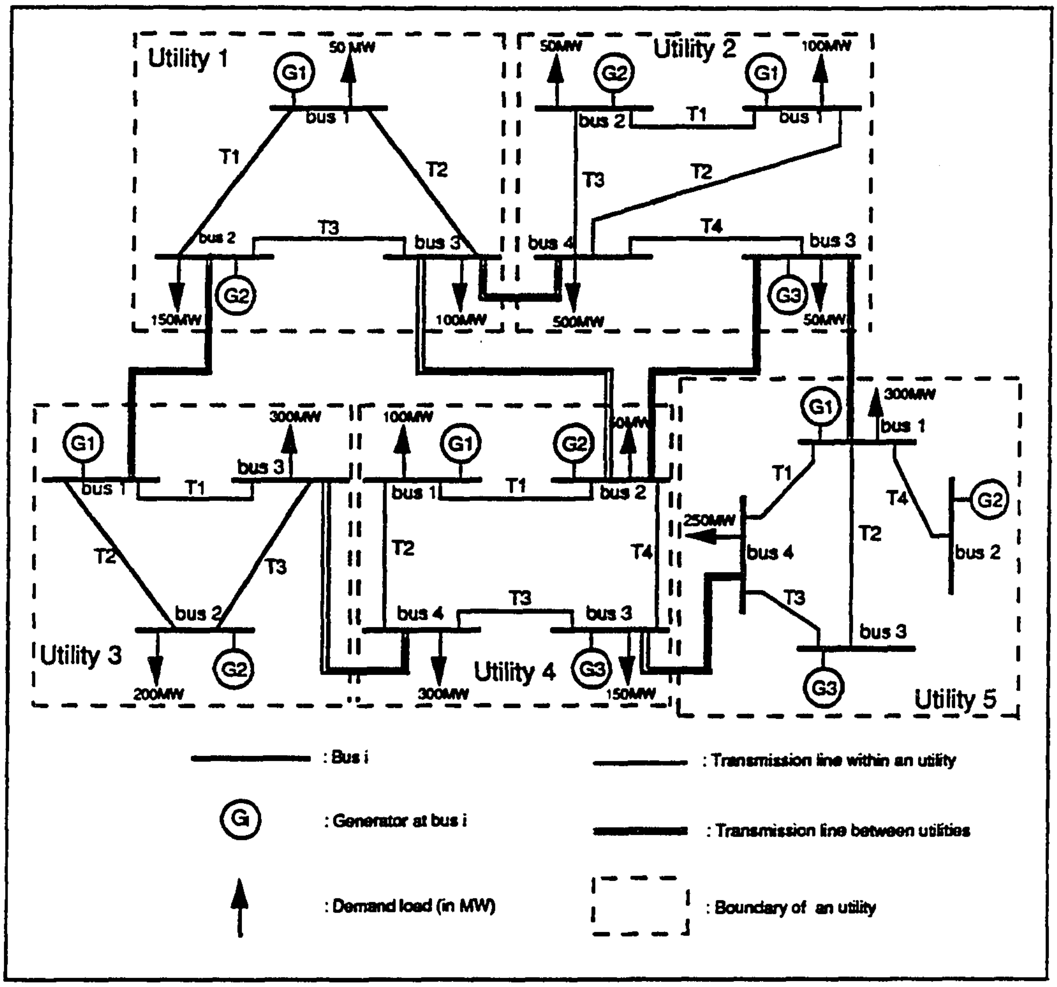

Figure 1. Five-utility interconnected electric power system

3. Basic Model for the Two-Stage Brokerage System

In this section, we will present the basic model for the two-stage brokerage system. Specifically, we will show: 1) how to determine the bids for buyers and sellers, 2) how to match the bids from buyers and sellers, 3) how to determine the transmission service charges for the intermediate transmission utilities, and 4) how to choose the route(s) to transmit the electric power. One assumption we make on the transportation-type economic dispatch model is that the linear terms in the objective function (2) (i.e., the production cost terms) are dominant relative to the non-linear terms in the objective function (2) (i.e., the transmission loss cost terms). This is reasonable when the cost of transmission loss is relatively small.

3.1 Determination of Bids for Buyers and Sellers

In this subsection, we show how the buyers and sellers determine their purchase and sale bids. Because of our assumption that the linear terms of the objective function (2) are dominant, the optimal strategies under the transportation-type economic dispatch model dictate the utilities produce power to the upper limit at the generator with smaller incremental costs (see e.g., Fahd, Richards, and Sheble [3]). Consequently, some extra generation capacities will exist at the generator with the highest incremental cost within each utility. Hence, the utility can produce power to the upper limit at the generator with the highest incremental cost and sell the surplus power to other utilities. On the other hand, the utility can shut down the generator with the highest incremental cost and purchase from other utilities the amount of power that the shut-down generator produces. For example, if the optimal level of the generation at the generator with the highest incremental cost and the upper limit of that generator are G_i^* and $\overline{G_i}$, then the possible sale quantity and purchase quantity of electric power for the utility are $\overline{G_i} - G_i^*$ and G_i^*, respectively. Also, we note that the purchase price and sale price for the electric power equal the incremental cost of the electric power being generated (see e.g., Fahd, Richards, and Sheble [3]).

Utility	MW to Buy	Purchase Price	Utility	MW to Sell	Sale Price
1	200	20	1	50	20
3	150	25	3	30	25
5	150	35	5	150	35

Table 2. Purchase and sale bids for buyers and sellers

<u>Example 2</u> (Continued from Example 1) Suppose only utility 1, utility 3, and utility 5 in the interconnected power system as shown in Fig. 1 can be buyers or sellers (i.e., only utility 2 and utility 4 can be intermediate transmission utilities).

The buying and selling bids for these utilities are as shown in Table 2.

3.2 The First Stage of the Brokerage System: Matching Bids from Buyers and Sellers

Once the central broker receives the bids from buyers and sellers, a linear programming model is set up to match the bids. At this stage, the brokerage system does not take the transmission service charge of the intermediate transmission utilities into account. We denote the per MW saving of the transaction between buyer j and seller i by ΔC_{ij}. The decision variable, the amount of transacted power between seller i and buyer j, is denoted by I_{ij}. The objective of this matching process is to maximize the total saving for all possible transactions subject to supply-demand constraints. Hence, we are concerned with the transactions with positive saving (i.e., $\Delta C_{ij} > 0$). Therefore, we mathematically formulate the matching process as the following linear programming.

$$\text{Maximize: } \Sigma\, I_{ij}\, \Delta C_{ij} \tag{6}$$

$$\text{subject to: } \sum_{j,j\neq i} I_{ij} \leq I_i^S \text{ for all i,} \tag{7}$$

$$\sum_{i,i\neq j} I_{ij} \leq I_j^P \text{ for all j,} \tag{8}$$

$$I_{ij} \geq 0, \text{ for all i and j.} \tag{9}$$

where I_i^S=sale quantity of seller i, and I_j^P=purchase quantity of buyer j.

Utility to Sell Power (i)	Utility to Buy Power (j)	Purchase Price (C_{Pj})	Sale Price (C_{Si})	Saving(ΔC_{ij})
1	3	25	20	5
1	5	35	20	15
3	5	35	25	10

Table 3. Positive cost coefficients of objective function in linear programming formulation.

Example 3 (Continued from Example 2) Let us perform the matching process for utility 1, utility 3, and utility 5 based on the results of Example 2.

The cost coefficients ('s) that are positive are shown in Table 3. The linear programming formulation for the matching process is now as follows.

$$\begin{aligned}
\text{Maximize:}\quad & 5\, I_{13} + 15\, I_{15} + 10\, I_{35} \\
\text{subject to:}\quad & I_{13} + I_{15} \leq 50 \\
& I_{35} \leq 30 \\
& I_{13} \leq 150 \\
& I_{15} + I_{35} \leq 150 \\
& I_{ij} \geq 0, \text{ for all i and j.}
\end{aligned}$$

The corresponding optimal solution is as follows: $I_{13} = 0$, $I_{15} = 50$, $I_{35} = 30$, and the total saving = 1050.

3.3 Calculation of Transmission Service Charges for Intermediate Transmission Utilities

After obtaining the outcomes of the matching process via linear programming, the central broker will inform the relevant intermediate utilities to provide transmission facilities and the corresponding transmission service charges. For example, utility 2 and utility 4 are the relevant intermediate utilities to the transactions among utility 1, utility 3, and utility 5 (see Figure 1). In this subsection, we discuss how an intermediate transmission utility determines his transmission service charge. Specifically, by treating the injected power and the extracted power due to the transaction as additional generations or additional loads, we can re-formulate the transportation-type economic dispatch model for the intermediate transmission utility. The transmission service charge (TSC) can be calculated from the difference in the total cost for the intermediate transmission utility with the transaction and without the transaction (see e.g., Shirmohammadi [7]). Namely,

$$TSC = \text{Total Cost with Transaction} - \text{Total Cost without Transaction}$$

<u>**Example 4**</u> (Continued from Example 3) Suppose the transaction between utility 1 and utility 5 employs the transmission facilities of utility 2. The configuration of the power system network is shown as Figure 1. Let us now determine the transmission service charge for utility 2.

For utility 2, we consider the transaction between utility 1 and utility 5 by treating bus 4 has an additional injected power generation of 50 MW and bus 3 has an additional extracted power load of 50 MW. By re-calculating the transportation-type economic dispatch, the optimal total cost with the transaction for utility 2 can be easily obtained as $18518.2. Therefore, the transmission service charge is equal to -$500 (= 18518.2 - 19018.2). The negative sign of the transmission service charge indicates the intermediate transmission utility is benefitted from providing transmission facilities.

3.4 The Second Stage of the Brokerage System: Selection of Route(s) to Transmit Electric Power

In the second stage of the brokerage system, after receiving the information of the transmission service charges from all relevant intermediate transmission utilities, the central broker will choose the route (or routes) with the minimum cost to transmit power. We assume that when there are multiple routes to transmit power for a transaction, the central broker will choose the one(s) with the least number of intermediate transmission utilities involved. It is reasonable because the more intermediate transmission utilities are involved, the more complex the transactions will physically become. When there are more than one route with the same least number of intermediate transmission utilities involved, the central broker will choose the one with the minimum transmission service charge. If the route with the minimum transmission service charge reaches its transmission capacity limit, then the central broker will choose the route with the second minimum transmission service charge, and so on. This process will be continued until all the transacted power has been transmitted, or all transmission capacity is exhausted. After the second stage of the brokerage system, the transactions among buyers, sellers, and intermediate transmission utilities are finalized.

Example 5 (Continued from Example 4) Let us determine the routes for the transactions from the matching process at the first stage of the brokerage system.

For the transaction between utility 1 and utility 5 (I_{ij} = 50MW), the transacted power can be transmitted via utility 2 or utility 4. Hence, the comparison of the transmission service charges from utility 2 and utility 4 is necessary. By considering the configuration of the electric power network shown in Figure 1 and employing the method described earlier for transmission service charges, we can have the transmission service charges shown in Table 4.

From Table 4, it is noted that the central broker will choose utility 2 as the intermediate transmission utility for the transaction between utility 1 and utility 5 due to the smaller transmission service charge.

For the transaction between utility 3 and utility 5, there is only

Transmit Power	with Transaction	without Transaction	Transmission Service Charge
via Utility 2	18518.2	19018.2	-500
via Utility 4	7356.0	7114.0	242

Table 4. Transmission service charges for intermediate transmission utilities.

one route (i.e., via utility 4) to transmit power from utility 3 to utility 5. For utility 4, the total costs with transaction and without transaction are $7057.84 and $7114.0; therefore, the transmission service charge is -$56.16.

4. Concluding Remarks

In this paper, a two-stage brokerage system for electric power transactions in an interconnected power system is presented. In the first-stage of the brokerage system, a linear programming model is set up to maximize the total saving from all potential transactions. In the second stage of the brokerage system, a method is presented to find the route(s) with the minimum transmission service charge for all transactions.

There are several possible extensions that will further improve the model presented in this paper. These extensions include incorporation of more sophisticated features of power systems such as the voltage, phase angle, and security issues into the interconnected power system.

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Appendix

The relevant data of the interconnected power system shown in Figure 1.

		Incremental Cost($/MW) or Resistance(1/MW)	Upper Limit (MW)	Lower Limit (MW)
Utility 1 (h=15)	Generator 1	12	100	0
	Generator 2	20	250	0
	Line 1	0.001	100	0
	Line 2	0.002	100	0
	Line 3	0.001	100	0
Utility 2 (h=20)	Generator 1	30	300	0
	Generator 2	25	350	0
	Generator 3	15	200	0
	Line 1	0.0015	150	0
	Line 2	0.003	200	0
	Line 3	0.001	400	0
	Line 4	0.002	250	0
Utility 3 (h=20)	Generator 1	10	350	0
	Generator 2	25	180	0
	Line 1	0.005	200	0
	Line 2	0.001	500	0
	Line 3	0.001	300	0
Utility 4 (h=12)	Generator 1	8	250	0
	Generator 2	10	200	0
	Generator 3	15	250	0
	Line 1	0.001	200	0
	Line 2	0.001	400	0
	Line 3	0.001	300	0
	Line 4	0.002	200	0
Utility 5 (h=25)	Generator 1	25	250	0
	Generator 2	10	150	0
	Generator 3	35	300	0
	Line 1	0.001	250	0
	Line 2	0.001	200	0
	Line 3	0.001	250	0
	Line 4	0.002	300	0

CHAPTER VI.

A TRILATERAL BROKERAGE SYSTEM FOR POWER TRANSACTIONS

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Summary

In this paper, we design and analyze a brokerage system for buyers, sellers, and intermediate utilities of electric power. Specifically, we mathematically characterize the determination of bids by buyers and sellers, the matching process of bids, and the selection of the transmission routes by the brokerage system. Moreover, we analyze the cost/benefit to intermediate utilities from the transmission of transacted power. The two key features differentiating this model from the extant literature on electric power transmission pricing and brokerage systems are: (1) multiple purchase and sale bids from potential buyers and sellers and (2) the systematic determination of transmission routes from minimizing the total cost to intermediate utilities. The improvement in economic efficiency (measured in terms of cost savings) is demonstrated via a series of numerical examples.

1. Introduction

The electric power industry in the United States is currently facing a drastic transformation from a traditional, regulated, and vertically integrated environment to a de-regulated and competitive environment (McCalley and Sheble ,1994). A primary motivation for this transformation is to improve the economic efficiency in the power industry. A critical research area where the power industry can improve the economic efficiency is that of power interchange in an interconnected power system. The power interchange may improve the economic efficiency because there exist some potential savings whenever the difference in incremental production costs among utilities is significant and excess production capacities exist. For the power interchange transactions among utilities, Chen and Min (1995) presented a two-stage brokerage system to match purchase bids and sales bids as well as to select the route(s) to transmit electric power. In this paper, by extending Chen and Min (1995) to allow multiple purchase bids and sale bids from each potential buyer and seller and by formulating the problem of selecting route(s) to transmit electric power as a nonlinear program, we show that the economic efficiency of the brokerage system for power transactions can be significantly improved.

The purpose of a brokerage system is to maximize the total benefit (saving) by matching the bids from buyers and sellers. The conventional brokerage system (see Doty and McEntire, 1982; or Fahd, Richards, and Sheble, 1992. for details) matches the highest purchase bid with the lowest sale bid, the second highest purchase bid with the second lowest sale bid, and so on. The matching process terminates when a viable match no longer exists. Doty and McEntire (1982) proposed two algorithms to improve the conventional brokerage systems: one employed a network flow algorithm and the other utilized dynamic programming techniques. Fahd, Richards, and Sheble (1992) implemented an energy brokerage system by employing linear programming. In their model, buyers and sellers can use the transmission networks of intermediate utilities and pay for the transmission service charges. The transmission service charges of their model are assumed to be strictly positive. This assumption, however, is not universally accepted. That is, the transmission service charges may be positive or negative (see e.g., Li and David, 1994). This implies that the transmission service charges, in their model, do not reflect the true cost/benefit of the intermediate transmission utilities. In contrast to Fahd, Richards, and Sheble (1992), Chen and Min (1995) proposed a two-stage brokerage system for power transactions so as to accurately reflect the true cost/benefit of the intermediate transmission utilities by considering physical aspects of the transmission such as the prevailing direction of power flow. In Chen and Min (1995), at the first stage, the brokerage system matches purchase bids and sale bids from buyers and sellers. Specifically, each buyer (seller) is restricted to have a single purchase (sale) bid. At the second stage, the brokerage system determines the route(s) to transmit the transacted power by some pre-specified rules.

In this paper, the two-stage trilateral brokerage system for power transactions is improved and extended as follows.

1) At the first stage, Chen and Min (1995) restricted each potential buyer or seller can only submit a single purchase bid and a single sale

bid. In this paper, multiple purchase bids and multiple sales bids are allowed for the buyer and the seller to submit to the central broker. The option of multiple purchase bids and multiple sale bids from the buyer and the seller may result in increased total cost saving because the case of a single purchase bid and a single sale bid from each buyer and each seller is a subset (or a special case) of the case that multiple purchase bids and sale bids are allowed.

2) At the second stage, Chen and Min (1995) presented some specified rules for the central broker to transmit the transacted power. That is, when there are multiple routes to transmit power for a transaction, the central broker will choose the one(s) with the least number of intermediate transmission utilities involved. When there are more than one route with the same least number of intermediate transmission utilities involved, the central broker will choose the one with the minimum transmission service charge. If the route with the minimum transmission service charge reaches its transmission capacity limit, then the central broker will choose the route with the second minimum transmission service charge, and so on. In contrast to Chen and Min (1995), instead of employing these pre-specified rules, we mathematically formulate the problem of selecting route(s) to transmit the transacted power as a nonlinear program. The objective now becomes to minimize the sum of the total cost of the intermediate transmission utilities involved in providing transmission facilities subject to the supply-demand relations at each bus and the capacity limits of each generator and each transmission line of the intermediate transmission utilities. In such a

case, the optimal solution for selecting route(s) to transmit the transacted power can be obtained.

By incorporating the above two aspects and by employing a numerical example, we will show that the economic efficiency of the brokerage system for power transaction can be significantly improved.

The following assumptions are made for the model:

1) Intermediate transmission utilities are neither buyers nor sellers.
2) The transmission service charges are small relative to the total savings from power interchange transactions.
3) The electric power flow can be treated as a commodity that can be transported by any selected transmission route subject to capacity restrictions (with advanced transmission systems such as the flexible AC transmission system (FACTS), it is a reasonable assumption, see e.g., Li and David, 1994).

The rest of this paper is organized as follows. First, we briefly review an economic dispatch model employing a transportation method. Then, we will show how the two-stage trilateral brokerage system is constructed. Also, in order to elucidate the two-stage trilateral brokerage system, several numerical examples are provided. Finally, the concluding remarks are presented.

2. Review of a Transportation-Type Economic Dispatch Model

The conventional economic dispatch (Wood and Wollenberg, 1984) concerns with the minimization of production cost subject to demand-supply relations and generation capacity constraints for an electric utility.

The optimal solutions, however, do not specify the power flow direction on each transmission line. In order to rectify this shortcoming and specify the power flow directions, Lee, Thorne, and Hill (1980) proposed an alternative economic dispatch model employing a transportation method. Our model will utilize this transportation-type economic dispatch model in determining bids for buyers and sellers as well as transmission service charges for intermediate transmission utilities. Hence, we first briefly review the transportation-type economic dispatch model. We employ the same notations as in Lee, Thorne, and Hill (1980) for an I-generator, N-bus, M-line system.

$I(n)$ = number of generators connected to bus n,

$M(n)$ = number of lines connected to bus n,

G_i = MW produced at generator i,

$F_i(G_i)$= the production cost for G_i MW at generator i,

D_n = MW load at bus n,

R_m = the resistance of transmission line m, measured in 1/MW

T_m = MW transmitted on line m,

The subscripts i, n, m are dummy counters for I, N, M respectively. Also, in Lee, Thorne, and Hill (1980), the transmission loss is directly related to the amount of power on a transmission line and can be approximately expressed by the following relation (see Elgerd, 1971. for details).

$$P_{Lm} = R_m T_m^2 \tag{1}$$

where P_{Lm} is the transmission loss on line m. Therefore, the total transmission loss in the system, $P_L = \sum^{M} R_m T_m^2$. Now, if we denote the marginal cost for transmission loss by h, then the cost function for transmission loss in the system will be hP_L.

Under these definitions and assumptions, the economic dispatch problem can be mathematically formulated as follows.

$$\text{Minimize: } F = \sum^{I} F_i(G_i) + hP_L \tag{2}$$

$$\text{subject to : } \sum^{I(n)} G_i - D_n + \sum^{M(n)} T_m = 0 \tag{3}$$

$$\underline{G_i} \leq G_i \leq \overline{G_i} \quad \text{for all } i, \tag{4}$$

$$0 \leq |T_m| \leq \overline{T_m} \quad \text{for all } m. \tag{5}$$

It is noted that the decision variables in the above system are G_i's and T_m's. Equation (3) represents the law of conservation at each bus (i.e., flow into the bus = flow out of the bus). Bar under and bar over represent lower and upper limits on the decision variables. Also, in this paper, we assume that the production cost function $F_i(G_i)$ has a linear relation with respect to the generation output G_i (see e.g., Fahd, Richards, and Sheble, 1992; or Wood & Wollenberg, 1984). We will first introduce the following example to illustrate the transportation-type economic dispatch model. Also, this example will be further utilized throughout this paper to illustrate the features of the two-stage brokerage system for electric power transactions.

Example 1 We now consider the five-utility interconnected electric power system shown in Figure 1. The relevant information of the generators and the transmission lines for each utility is shown in Appendix. The optimal solution for each utility by employing the transportation-type economic dispatch is shown in Table 1.

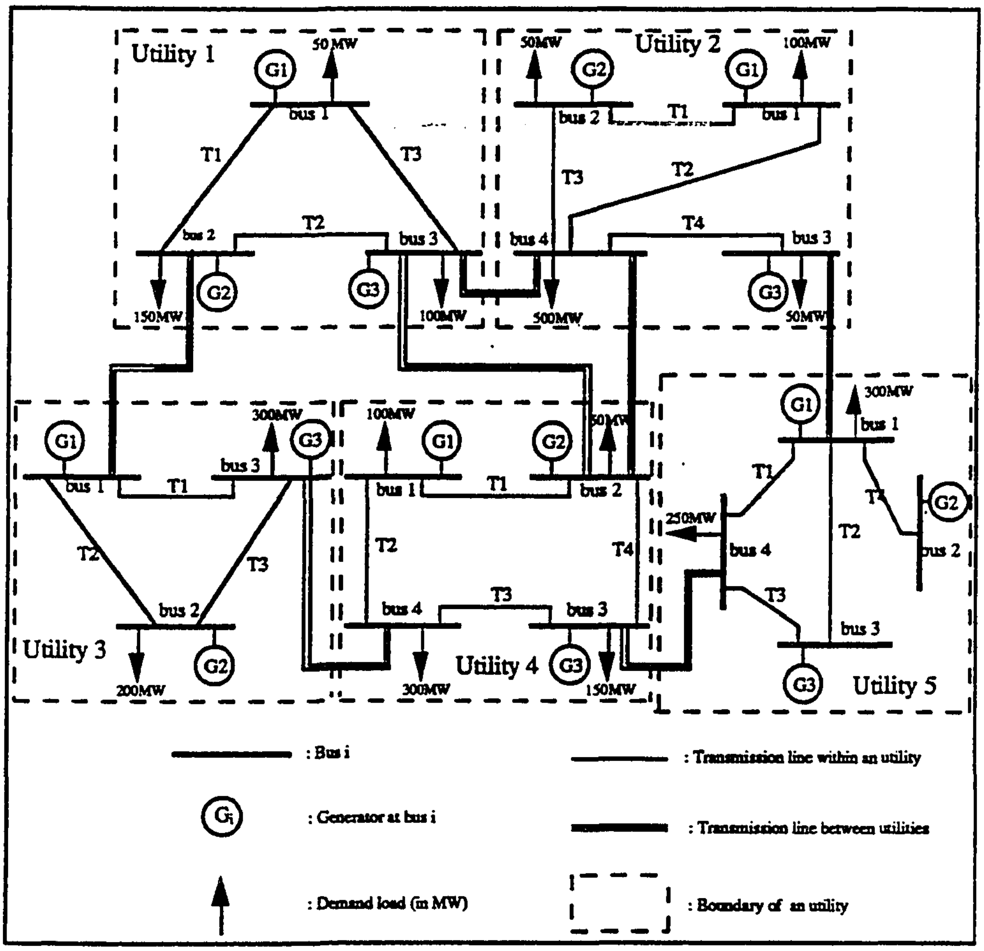

Figure 1. Five-utility interconnected electric power system

	G_1	G_2	G_3	T_1	T_2	T_3	T_4	Cost
Utility 1	100	200	0	12.5	37.5	62.5	-	3303.1
Utility 2	150	350	200	27.3	77.3	272.7	150	19018.2
Utility 3	350	150	0	175.0	175.0	125.0	-	9400.0
Utility 4	250	200	150	60.0	210.0	90.0	90	7114.0
Utility 5	250	150	150	116.7	16.7	133.3	150	14916.7

Table 1. Optimal solution from transportation-type economic dispatch model.

3. The Trilateral Brokerage System

In this section, we will present the basic model for the two-stage trilateral brokerage system. Specifically, we will show: 1) how the buyers and the sellers determine their purchase bids and sale bids, 2) how the brokerage systems matches the purchase bids from buyers and sale bids from sellers, 3) how the brokerage system chooses the route(s) to transmit the transacted electric power, and 4) how the intermediate transmission utilities can be benefited or cost by providing transmission facilities. One assumption we make on the transportation-type economic dispatch model is that the linear terms in the objective function (2) (i.e., the production cost terms) are dominant relative to the non-linear terms in the objective function (2) (i.e., the transmission loss cost terms). This is reasonable when the cost of transmission loss is relatively small.

3.1 Determination of Bids for Buyers and Sellers

In this subsection, we show how the buyers and sellers determine their purchase and sale bids under the assumption that multiple purchase bids and sales bids are allowed to submit to central broker from each potential buyer or seller. Because of our assumption that the linear terms of the objective function (2) are dominant, the optimal strategies under the transportation-type economic dispatch model dictate the utilities produce power to the upper limit at the generator with smaller incremental costs (see e.g., Fahd, Richards, and Sheble, 1992). Consequently, some extra generation capacities will exist at the generators with the higher incremental cost within each utility. Hence, the utility can produce power to the upper limits at the generators with the higher incremental costs and sell the surplus power to other utilities. On the other hand, the utility can shut down the generators with the lower incremental costs and purchase from other utilities the amount of power that the shut-down generators produce. For example, suppose there are five generators within an electric utility, and the optimal levels of these five generators are $G_1^* = \overline{G_1}$, $G_2^* = \overline{G_2}$, $G_3^* = \hat{G}_3$ (where $\underline{G_3} \leq \hat{G}_3 \leq \overline{G_3}$), $G_4^* = G_5^* = 0$, respectively. Then the possible sale quantities for sale bids are $\overline{G_3} - \hat{G}_3$, $\overline{G_4}$, and $\overline{G_5}$. On the other hand, the possible purchase quantities for purchase bids are $\hat{G}_3$, $\overline{G_2}$, and $\overline{G_1}$. Also, we note that the purchase price and sale price for the electric power equal the incremental cost of the electric power being generated (see e.g., Fahd, Richards, and Sheble, 1992).

Purchase Bids			
Utility	Bids	MW to Purchase	Purchase Price
1	1st	200	12
1	2nd	100	8
3	1st	150	20
3	2nd	350	10
5	1st	150	35
5	2nd	250	25
5	3rd	150	10
Sale Bids			
Utility	Bids	MW to Sell	Sale Price
1	1st	50	12
1	2nd	100	18
3	1st	100	20
3	2nd	150	30
5	1st	150	35

Table 2. Purchase and sale bids for buyers and sellers.

Example 2 (Continued from Example 1) Suppose only utility 1, utility 3, and utility 5 in the interconnected power system as shown in Fig. 1 can be buyers or sellers (i.e., only utility 2 and utility 4 can be intermediate transmission utilities). Then the buying and selling bids for these utilities are as shown in Table 2.

We note that, if we allow single purchase bid and single sale bid in this example, there are only three purchase bids and three sale bids.

3.2 Matching Bids from Buyers and Sellers: Formulation of Linear Programming

Once the central broker receives the bids from buyers and sellers, a linear programming model is set up to match the purchase bids and sale bids. At this stage, the brokerage system does not take the transmission service charge of the intermediate transmission utilities into account. We denote the per MW saving of the transaction between buyer b's ith purchase bid and seller s's jth sale bid by $\Delta C_{bi,sj}$. The decision variable, the amount of transacted power between seller s's jth sale bid and buyer b's ith purchase bid is denoted by $I_{bi,sj}$. The objective of this matching process is to maximize the total saving for all possible transactions subject to supply-demand constraints. Hence, we are only concerned with the transactions with positive saving (i.e., $\Delta C_{bi,sj} > 0$). Therefore, for the terms corresponding to positive $\Delta C_{bi,sj}$'s only, we mathematically formulate the matching process as the following linear programming.

$$\text{Maximize:} \quad \sum_{b,i,s,j} I_{bi,sj}\, \Delta C_{bi,sj} \tag{6}$$

$$\text{subject to:} \quad \sum_{b,i} I_{bi,sj} \leq I_{sj} \quad \text{for all s and j,} \tag{7}$$

$$\sum_{s,j} I_{bi,sj} \leq I_{bi} \quad \text{for all b and i,} \tag{8}$$

$$I_{bi,sj} \geq 0, \text{ for all b, s, i and j.} \tag{9}$$

where I_{sj} is the total sale quantity for seller s's jth bid and I_{bi} is the total purchase quantity for buyer b's ith bid.

Example 3 (Continued from Example 2) Let us perform the matching process for utility 1, utility 3, and utility 5 based on the results of Example 2.

Utility to Sell Power		Utility to Buy Power		Purchase Price	Sale Price	Saving
Utility	Bids	Utility	Bids			
1	1st	3	1st	20	12	8
1	1st	5	1st	35	12	23
1	1st	5	2nd	25	12	13
1	2nd	3	1st	20	18	2
1	2nd	5	1st	35	18	17
1	2nd	5	2nd	25	18	7
3	1st	5	1st	35	20	15
3	1st	5	2nd	25	20	5
3	2nd	5	1st	35	30	5

Table 3. Positive cost coefficients of objective function in linear programming.

The cost coefficients (’s) that are positive are shown in Table 3. The linear programm for the matching process is now as follows.

Maximize: $8\ I_{31,11} + 23\ I_{51,11} + 13\ I_{52,11} + 2\ I_{31,12} + 17\ I_{51,12}$
$+ 7\ I_{51,12} + 15\ I_{51,31} + 5\ I_{52,31} + 5\ I_{51,32}$

subject to: $I_{31,11} + I_{51,11} + I_{52,11} \leq 50$

$I_{31,12} + I_{51,12} + I_{52,12} \leq 100$

$I_{51,31} + I_{52,31} \leq 30$

$I_{51,32} \leq 150$

$I_{31,11} + I_{31,12} \leq 150$

$I_{51,11} + I_{51,12} + I_{51,31} + I_{51,32} \leq 150$

$I_{52,11} + I_{52,12} + I_{52,31} \leq 250$

$I_{bi,sj} \geq 0$, for all b, s, i and j.

The corresponding optimal solution is as follows: $I_{31,11} = I_{52,11} = I_{31,12} = I_{52,12} = I_{51,31} = I_{51,32} = 0$, $I_{51,11} = 50$, $I_{51,12} = 100$, and $I_{52,31} = 30$. That is, utility 5 will purchase 150 MW from utility 1 and 30 MW from utility 3, and the total cost saving is $3000. We note that, given the same model environments in this example, if only single purchase bid and single sale bid are allowed from the potential seller and buyer to submit to central broker, the result is: utility 5 purchases 50 MW from utility 1 and 30 MW from utility 3, and the total cost saving is $1700 (see Chen and Min, 1995, for details). From this comparison, by allowing multiple purchase bids and multiple sale bids from each potential seller and buyer, the total cost saving is increased by $1300 which is approximately 76.5% (= (3000-1700)/1700).

3.3 Selection of Route(s) to Transmit Electric Power

After obtaining the outcomes of the matching process via linear programming, the central broker will inform the relevant intermediate utilities to provide transmission facilities. For example, utility 2 and utility 4 are the relevant intermediate transmission utilities to the transactions among utility 1, utility 3, and utility 5 (see Figure 1). In this subsection, we discuss how the central broker chooses the route (or routes) to transmit the transacted power. We assume that the central broker has the complete information of the relevant intermediate transmission utilities such as incremental cost at each generator, resistance at each transmission line, etc. Also, we note that the injected power and the extracted power due to the transaction can be treated as additional generations or addition loads for the intermediate transmission utility. For the central broker, the objective is to minimize the sum of the total costs of intermediate transmission utilities. In order to mathematically formulate the problem, we first introduce the following notations.

G_{ij} = the generation output at generator j of utility i.

T_{im} = MW transmitted at transmission line m of utility i.

X_{sb}^{ITU} = MW transmitted from seller s to buyer b via ITU (intermediate transmission utilities)

Therefore, the objective function can be formulated as follows.

$$\text{Minimize:} \quad \sum_{i \in ITU} TC_i = \sum_i \left[\sum_j F_{ij}(G_{ij}) + h_i \sum_m R_{im} T_{im}^2 \right] \tag{10}$$

Also, the conservation law at bus j (i.e., flow in = flow out at each bus) within utility i can be expressed as follows.

$$\Sigma\, G_{ij} - D_{in} + \Sigma\, T_{im} + \Sigma\, X_{sb}^{ITU} = 0 \text{ for all bus j within utility i} \tag{11}$$

We should also consider the generation capacity limits at each generator as well as the transmission capacity limits at each transmission line. The total amount of transacted power from seller s to buyer b, I_{sb}, should be equal to or greater than the sum of the MW transmitted from seller s to buyer b through all different intermediate transmission utilities (i.e., $\sum_{ITU} X_{sb}^{ITU}$). Therefore, the complete mathematical formulation for the central broker to choose the route(s) is as follows.

$$\text{Minimize:} \quad \sum_{i \in ITU} TC_i = \sum_i \left[\sum_j F_{ij}(G_{ij}) + h_i \sum_m R_{im} T_{im}^2 \right] \tag{12}$$

subject to:

$$\Sigma G_{ij} - D_{in} + \Sigma T_{im} + \Sigma X_{sb}^{ITU} = 0 \text{ for all bus j of utility i} \tag{13}$$

$$\sum_{ITU} X_{sb}^{ITU} = I_{sb} \text{ for all transactions} \tag{14}$$

$$\underline{G_{ij}} \le G_{ij} \le \overline{G_{ij}} \text{ for all i and j} \tag{15}$$

$$0 \le |T_{im}| \le \overline{T_{im}} \text{ for all i and m.} \tag{16}$$

We note that the decision variables in the model are X_{sb}^{ITU}'s and T_{im}'s.

__Example 4__ (Continued from Example 3) Select the routes to transmit the transacted power of 150 MW from utility 1 to utility 5 and 30 MW from utility 3 to utility 5.

According to Figure 1 and the result of matching process from Example 3, we can have the following mathematical formulation for this problem.

$$\begin{aligned}\text{Minimize } & 30G_{21}+25G_{22}+15G_{23}+ 20(0.0015T_{21}^2 + 0.003T_{22}^2 + 0.001T_{23}^2 + \\ & 0.002T_{24}^2) +8G_{41}+10G_{42}+15G_{43}+ 12(0.001T_{41}^2 + 0.001T_{42}^2 + \\ & 0.001T_{43}^2 + 0.002T_{44}^2)\end{aligned}$$

$$\begin{aligned}\text{subject to : } & G_{21} - 100 + T_{21} - T_{22} = 0 \\ & G_{22} - 50 - T_{21} - T_{23} = 0 \\ & G_{23} - 50 - T_{24} - X_{15}^2 - X_{35}^{42} = 0 \\ & T_{22} + T_{23} + T_{24} - 500 + X_{15}^2 + X_{35}^{42} = 0 \\ & G_{41} - 100 + T_{41} - T_{42} = 0 \\ & G_{42} - 50 - T_{41} - T_{44} + X_{15}^4 - X_{35}^{42} = 0 \\ & G_{43} - 150 + T_{44} - T_{43} - X_{15}^4 - X_{35}^4 = 0 \\ & T_{42} + T_{43} - 300 + X_{35}^4 + X_{35}^{42} = 0\end{aligned}$$

$$X_{15}^{2} + X_{15}^{4} \le 150$$
$$X_{35}^{4} + X_{35}^{42} \le 30$$
$$\underline{G_{ij}} \le G_{ij} \le \overline{G_{ij}} \text{ for } i = 1, 2, 3, 4 \text{ and } j = 2, 4$$
$$0 \le |T_{im}| \le \overline{T_{im}} \text{ for } i = 2, 4 \text{ and } m = 1, 2, 3, 4.$$

By employing GINO (an mathematical optimization software), we can easily obtain the optimal solution as follows: $X_{15}^{2} = 150$, $X_{35}^{42} = 30$, $X_{15}^{4} = 0$, and $X_{35}^{4} = 0$. That is, 150 MW from utility 1 to utility 5 will utilize the transmission facilities of utility 2 while 30 MW from utility 3 to utility 5 will utilize the transmission facilities of utility 4 and utility 2. If we employ the pre-specified rules developed in Chen and Min (1995), the result of selecting routes to transmit the transacted power is to utilize the transmission facilities of utility 2 to transmit 150 MW from utility 1 to utility 5 (i.e., $X_{15}^{2} = 150$) and utilize the transmission of utility 4 to transmit 30 MW from utility 3 to utility 5 (i.e., $X_{35}^{4} = 30$).

3.4 Cost/Benefit Calculation for Intermediate Transmission Utilities

The transmission service charge (TSC) can be calculated from the difference in the total cost for the intermediate transmission utility with the transaction and without the transaction (see e.g., Shirmohammadi *et al.*, 1991). Namely,

TSC = Total Cost with Transaction - Total Cost without Transaction

Example 4 (Continued from Example 3) Calculate the transmission service charges for utility 2 and utility 4.

For utility 2, we consider: 1) the transaction between utility 1 and utility 5 by treating bus 4 has an additional injected power generation of 150 MW and bus 3 has an additional extracted power load of 150 MW; and 2) the transaction between utility 3 and utility 5 by treating bus 4 has an additional injected power generation of 30 MW and bus 3 has an additional extracted power load of 30 MW. By re-calculating the transportation-type economic dispatch, the optimal total cost with the transactions for utility 2 can be easily obtained as \$18154.2. Therefore, the transmission service charge for providing transmission facilities for utility 2 is equal to -\$864 (= 18154.2 - 19018.2). The negative sign of the transmission service charge indicates the intermediate transmission utility is benefited from providing transmission facilities.

For utility 4, the total costs with transaction and without transaction are \$6932.56 and \$7114. Therefore, the transmission service charge is -\$181.44.

We note that, from the perspective of the central broker, the total sum of the transmission service charges for the brokerage system is -\$1045.44 (= -864 -181.44). Under the same model environments, if we employ the pre-specified rules proposed in Chen and Min (1995), the total sum of the transmission service charges is -\$956.16. The economic efficiency is improved approximately 9.33% (= (1045.44-956.16)/956.16).

4. Concluding Remarks

In this paper, we extended the model of the two-stage trilateral brokerage system discussed in Chen and Min (1995) to the following two aspects. First, the restriction of single purchase bid and single sale bid from each potential buyer and seller is relaxed. In this paper, we allow multiple purchase bids and multiple sale bids from each potential buyer and seller. By formulating a linear program to maximize the total cost saving in matching bids from sellers and buyers, we show that the total cost saving can be significantly improved under the assumption that multiple purchase bids and multiple sale bids are allowed. Second, instead of employing the pre-specified rules proposed in Chen and Min (1995) to determine the route(s) to transmit the transacted power , we mathematically formulate the problem of selecting routes to transmit the transacted power as a nonlinear program. In such a case, the optimal solution for selecting route(s) to transmit the transacted power can be obtained.

There are several possible extensions that will further improve the model presented in this paper. These extensions include incorporation of more sophisticated features of power systems such as the voltage, phase angle, and security issues into the interconnected power system.

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Appendix

The data information of the interconnected power system shown in Figure 1.

		Incremental Cost($/MW) or Resistance(1/MW)	Upper Limit (MW)	Lower Limit (MW)
Utility 1 (h=15)	Generator 1	8	100	0
	Generator 2	12	250	0
	Generator 3	18	100	0
	Line 1	0.001	100	0
	Line 2	0.002	100	0
	Line 3	0.001	100	0
Utility 2 (h=20)	Generator 1	30	300	0
	Generator 2	25	350	0
	Generator 3	15	200	0
	Line 1	0.0015	150	0
	Line 2	0.003	200	0
	Line 3	0.001	400	0
	Line 4	0.002	250	0
Utility 3 (h=20)	Generator 1	10	350	0
	Generator 2	20	180	0
	Generator 3	30	150	0
	Line 1	0.002	200	0
	Line 2	0.001	500	0
	Line 3	0.001	300	0
Utility 4 (h=12)	Generator 1	8	250	0
	Generator 2	10	200	0
	Generator 3	15	250	0
	Line 1	0.001	200	0
	Line 2	0.001	400	0
	Line 3	0.001	300	0
	Line 4	0.002	200	0
Utility 5 (h=25)	Generator 1	25	250	0
	Generator 2	10	150	0
	Generator 3	35	300	0
	Line 1	0.001	250	0
	Line 2	0.001	200	0
	Line 3	0.001	250	0
	Line 4	0.002	300	0

GENERAL CONCLUSIONS

In this dissertation, we investigated how lot-size decision makers and electric power utilities determine critical economic quantities so as to improve the economic efficiency of operations. Throughout this dissertation, the optimal policies were obtained through linear and nonlinear programming techniques. For each model, interesting managerial insights and economic implications were obtained and illustrative numerical examples were provided. For each chapter of this dissertation, we present a detailed summary and possible extensions as follows.

In Chapter 1, we constructed and analyzed EOQ-type models for a buyer who was just informed of a temporary sale. For such a buyer, optimal inventory/disposal policies were derived by comparing cost savings of various cases. By analyzing the optimal inventory/disposal policies, several managerial insights were obtained. Several possible extensions can be made to enhance the inventory model developed in Chapter 1. For example, one class of extensions can be made with respect to the option of disposal. In Chapter 1, it is assumed that the seller will not react to the buyers' disposal (if any). It would be of interest to investigate several possible policies of a seller. e.g., prohibition of disposals, benefit sharing of disposals, etc. We believe that such extensions will improve the applicability in practice of the inventory/disposal models in response to sales. We hope this improvement in applicability will result in increased economic efficiency for the buyer (as well as the seller).

In Chapter 2, we constructed and analyzed an EOQ-type model for a buyer who is just informed of a pre-announced sale. By "a pre-announced sale", we mean the announcement time of the sale occurs before the beginning time of the sale. For such a buyer, optimal inventory policies are derived by comparing cost savings of various cases. By analyzing the optimal inventory policies, several managerial insights are obtained. Several possible extensions can be made to enhance the inventory models developed in Chapter 2. For Example, it is assumed that the sale period is less than one regular EOQ replenishment cycle. By relaxing this assumption and allowing the sale period is greater than one regular EOQ replenishment cycle, interesting models that augment the models in Chapter 2 can be developed.

In Chapter 3, we constructed and analyzed inventory and investment in setup cost operations models under profit maximization and return on investment maximization for lot-size decision makers. First, we showed how inventory and investment in setup operations models under profit maximization and return on investment maximization can be formulated for general functional form of the investment in setup operations. From these formulations, the optimality conditions and the corresponding economic interpretations are obtained. Next, for the specific cases of the linear setup cost and the hyperbolic setup cost, the optimal closed-form solutions are obtained and several interesting managerial insights are presented.

The models developed in Chapter 3 relates general practices since numerous industries and firms apply EOQ based decision making for their inventory systems. There are several possible extensions that will further improve the relevance of our models to general practices. They include incorporation of more sophisticated features such as shortages, delivery lags, and stochastic demand rates, etc. From the perspective of investing in setup operations, it would be of interest to study the allocation of the investment in setup operations. For example, how much should be invested in purchasing or leasing new equipments and how much should be invested in labor's training and wages, etc. From the perspective of optimization criterion, it would be of interest to study the effects of investing in setup operations on process quality improvement, effective capacity and flexibility improvement (see e.g., Porteus, 1986, and Spence and Porteus, 1987) in conjunction with the optimization criterion of return on investment.

In Chapter 4, we presented two major revisions/corrections regarding a recent paper by T. C. E. Cheng (1990) — "An EOQ Model with Pricing Consideration". First, we pointed out that the critical assumption of equal replenishment cycle length for each product was not incorporated into his model formulation. We reformulated the entire model and derived the corresponding Kuhn-Tucker conditions. Next, in the case of linear demand functions, we indicated that the solutions provided by Cheng (1990) may result in non-optimal solution, or multiple candidates. By employing the trigonometric methods, we derived the optimal closed-form solution that is unique and obtained the conditions under which the optimal closed-form solution is valid.

In Chapter 5, a two-stage brokerage system for electric power transactions in an interconnected power system is presented. In the first-stage of the brokerage system, a linear programming model is set up to maximize the total saving from all potential transactions. In the second stage of the brokerage system, a method is presented to find the route(s) with the minimum transmission service charge for all transactions.

Chapter 6 revised Chapter 5 in the following two aspects. First, multiple purchase bids and multiple sales bids are allowed for the buyers and the sellers to submit to the central broker. This may result in increased total cost saving because the case of a single purchase bid and a single sale bid from each buyer and each seller is a subset (or special case) of the case that multiple purchase bids and sale bids are allowed. Second, we mathematically formulate the problem of selecting route(s) to transmit the transacted power as a nonlinear program. In such a case, the solution for selecting route(s) to transmit the transacted power can be optimally obtained.

There are several possible extensions that will further improve the models presented in Chapter 5 and Chapter 6. These extensions include incorporation of more sophisticated features of power systems such as the voltage, phase angle, reactive power, and security issues into the interconnected power system.

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